A Method for the Automated Discovery of Angle Theorems

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The Naive Angle Method, used by Geometry Expressions for solving problems which involve only angle constraints, represents a geometrical configuration as a sparse linear system. Linear systems with the same underlying matrix structure underpin a number of different geometrical theorems. We use a graph theoretical approach to define a generalization of the matrix structure.

1 Introduction

One approach to Geometric Discovery starts with a given geometry diagram, and hunts systematically, or unsystematically for provable statements about the geometric entities, or further derived geometric entities [1]. The given diagram can be in fact a parametrized family of diagrams [5]. Another approach [4] is to start with the statements one wants to prove, and discover supplementary conditions required to make the theorems true. Again, however, the geometric milieu is given. A problem for such systems is to determine the interestingness of generated theorems, metrics for which are an active topic of research[2].

In this paper, we consider working in reverse and generating the geometric diagram to match a more abstract form of the theorem, which guarantees both its solution, but also a certain level of interestingness. The abstract form is developed by analogy with known theorems, considered (by this author) to be aesthetically pleasing. We develop and automate here a method for generating many theorems of comparable structure but different geometry to our seed theorems. Hopefully this might lend us some control of the richness and tractability, even aesthetic appeal of our generated theorems.

Having promised emergent geometry, we immediately limit the scope of our work, however, to consider theorems in the Naive Angle Method employed by Geometry Expressions [9] for angle specific problems. While the method accommodates a number of different constraint types, in the bulk of this paper, we focus solely on the angle bisector constraint, which can be disguised as an isosceles triangle, a circle chord, or a reflection. In any case, it contributes a row with 3 values -1,-1,2 to the constraint matrix. At one level, we can re-interpret the same matrix using different geometry: for example changing a circle chord into an angle bisector (figure 1). At another level, we consider matrices with non zero elements in the same places, but with different assignments within the row of the numerical values (they will still be -1, -1 , 2, only their order will be different). At a third level, we generalize to consider matrices with a similar pattern of non-zero positions. For a class of such matrices, we give structural conditions which determine the presence or absence of theorems of comparable interest to the prototype.

2 The Naive Angle Method

The Full Angle Method [3] treats an angle and its supplement as the same thing. It has the benefit of allowing theorems to be expressed in a more general form, and hence allowing multiple instances of

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the same theorem to be proved at once [7]. However, it does not allow theorems to be expressed which themselves depend on the conventional notion of an angle, as the difference in the direction of two signed rays. Geometry Expressions [9] employs the Analytical Geometry Method [8] to derive an expression for output measurements in terms of symbolic values for input constraints. The method relies on deriving a symbolic Cartesian coordinate description of the model. When an angle is measured, it is derived from the Cartesian equations of its lines using inverse trigonometric functions. Simplification of expressions involving such functions imposes a heavy burden on the algebra system, and results for angle-dominated diagrams are not satisfactory. This prompted the development of an auxiliary system which is deployed when the entire computation can be kept in the angle domain. For example, if a triangle is defined by one side and two angles, and the third angle is measured, the Cartesian computation involves an inverse trigonometric function. However, expressed in terms of angles the result is simply the difference between π and the sum of the other two angles. In contrast to the full angle method, this approach considers angles to be signed, and hence an angle and its supplement are different. This means that theorems are less general, but has the advantage of corresponding with the conventional notion of angle used by consumers of mathematics such as students and engineers.

Let $d_1 \dots d_n$ be the directions of the n (directed) lines comprising a geometric figure. A number of different constraints may be applied in Geometry Expressions, each of which may be expressed as a linear equation:

1. angle between line i and j is ϕ : $d_i - d_j = \phi$
2. line k bisects line i and line j : $2d_k - d_i - d_j = 0$
3. line k is the base of an isosceles triangle whose equal sides are i and j : $2d_k - d_i - d_j = \frac{\pi}{2}$
4. line j is the image of line i under reflection in j : $2d_k - d_i - d_j = 0$

Circles contribute to the angle model in two ways:

1. Tangents contribute right angles with the line joining the point of tangency to the center of the circle.
2. Chords contribute isosceles triangle relationships with their end point radii.

The linear equations above are all to be considered modulo 2π. When the resulting linear system is solved, symbolic directions are determined for each line, which can be subtracted to yield angles. Resolving how many 2π's to add or subtract from the result is done by preserving a numerical prototype of the geometry (a sketch) which is used to arbitrate this issue.

To solve for the value of an angle, the columns of the matrix are reordered so that the columns corresponding to the two lines which define the angle are at the right. Gaussian Elimination is then run forward to create an upper diagonal matrix. If the Gaussian Elimination runs to completion with non-zero values in the last two places of the final row and zeros in the rest then the angle is determined.

Any row constructed by Gaussian Elimination is in the null space of the row vectors representing the constraints on the problem. Establishing that a row representing a specific angle is in that null space is equivalent to proving that the value of the angle may be determined from the constraint equation. The actual value of the angle will be generated in the course of the Gaussian Elimination applied to an additional matrix column containing the right hand sides of the constraint equations.

3 Theorem Discovery

A statement in the Naive Angle Method is a linear combination of line directions. It can be proven true and constitutes a theorem if it is in the row space of the hypothesis vectors. If there are m rows (hypotheses) and n columns (geometric lines), then, assuming the hypotheses are linearly independent, they span a space of dimension m from a space of dimension $n-1$ (all rows satisfy the linear condition that the sum of their coefficients is 0). Let C be a set of $n-m+1$ or more columns of the matrix (or equivalently, geometric lines). Then there is a row vector in the span of the hypothesis vectors whose non-zero coefficients are all in C.

For any set C of sufficient size, then, a theorem exists. Some are more interesting than others. For a given diagram, we are most interested in theorems which use all the hypotheses: that is their vector does not belong to a space spanned by any subset of the hypothesis vectors. A theorem vector is less common, and thus more interesting, the fewer non-zero coefficients it contains. An algorithm for finding interesting theorems in a given diagrams takes the following form:

1. Augment the matrix M with a final column containing the symbols $r_1, \ldots r_m$
2. Iterate through a space of potential column sets C , either exhaustively, or using random permutations.
3. Reorder the matrix columns so that the final $n-m+1$ columns correspond to the indices in C.
4. Perform Gaussian Elimination to compute the upper triangular matrix U in the LU decomposition.
5. The final row of U contains the theorem in its first n entries and its expression as a linear combination of rows in the final column.
6. decide which 'theorems' to keep based on a heuristic which can use the number of hypotheses involved and the number of non zero coefficients in the theorem.

3.1 Matrix Structure of Seed Theorems

The approach above starts with a geometry diagram which then defined a matrix, we took the matrix and used it to generate quantities which were guaranteed to be true, then applied heuristics to determine which of these true facts was worth holding onto as a theorem. We'd like to consider starting with a matrix and generating the geometry diagram from the matrix. As there are three different constraints which contribute a row with values -1, -1, 2, such a matrix row can be interpreted in those three different ways. Further, isosceles triangle constraints may be represented by a line forming the chord of a circle.

Hence the same matrix can represent two apparently quite different diagrams. For example, in figure 1(a) E is the intersection of the diagonals of cyclic quadrilateral $ABCD$, F is the circumcenter of ADE, the theorem states that BC and EF are perpendicular [7]. Radial lines are added as needed in the Naive Angle Method and are shown dashed in the diagram.

Numbers on the diagram reference matrix columns. The corresponding matrix is shown below

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Figure 1b shows a second theorem which has the same matrix. $ABCD$ is a general quadrilateral. The angle bisector at C intersects the angle bisectors at B and D in E and F, while the angle bisector at B intersects the angle bisector at D in G. H is the circumcenter of EFG. The theorem states that HG is

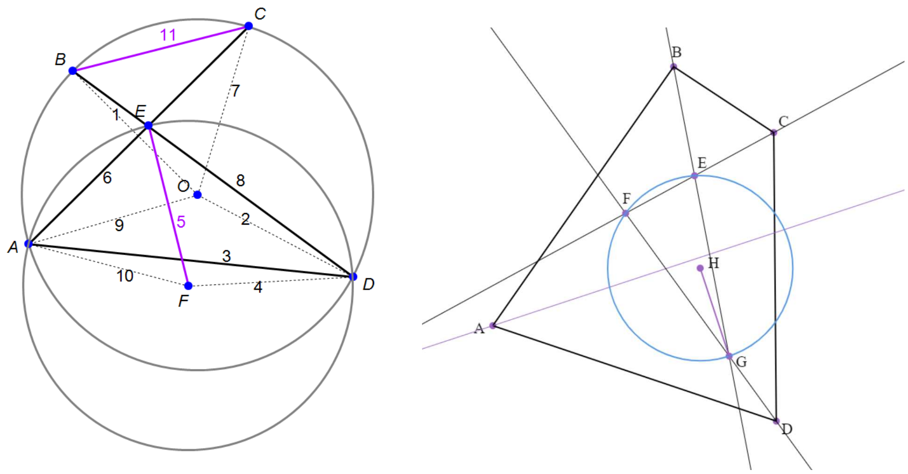

Figure 1: Two theorems which share the same matrix

perpendicular to the bisector of angle A. Rows 1,2,3,6 are interpreted in figure 1b as angle bisectors, while in figure 1a they are considered to be isosceles triangle constraints.

3.2 Different Matrices with the Same Shape

Figure 2a states that if $BCDE$ is a cyclic quadrilateral and F lies on EB extended while G lies on DC extended, and $BFGC$ is also cyclic, then FG and ED are parallel [7]. Radial lines, added automatically in the Naive Angle Method, are shown dashed. Numbers in the diagram correspond to columns in the matrix.

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2
\end{bmatrix}$$

Figure 2b illustrates the following theorem from geometrical optics. The image of parallel rays reflecting at points A and C in the sides AB and BC of triangle ABC lies on the circumcircle of A, B and O, where O is the circumcenter of ABC. With the columns numbered as given in the figure, the matrix for this diagram is as follows:

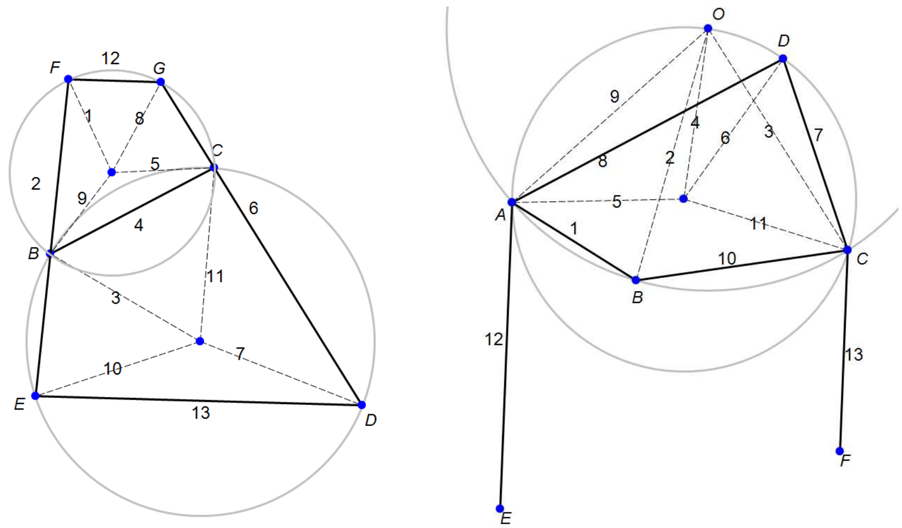

Figure 2: Two theorems whose matrices have non-zero elements in the same locations, but which are not identical

$$\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1
\end{bmatrix}$$

Examining these matrices, we see that both have non-zero elements in the same positions of the matrix, however the values in those elements are not identical. The matrices have the same shape, but are not identical.

As the final two columns in this matrix have only one entry, any theorem which uses all hypotheses must incorporate these two columns. At this point, we narrow our focus to theorems whose statement as a linear combination of line directions contain only two non-zero coefficients. As the sum of each row of our matrix is 0, then a theorem with 2 non zero coefficients must specify a scalar multiple of the difference between the line directions, or the angle between the lines.

With this narrower focus, matrices of this shape which yield theorems can be determined as follows:

1. Iterate through all 3^8 different locations for the 2 in each row.
2. Perform Gaussian Elimination to determine the upper triangular component of the LU decomposition.
3. Examine its last row: if it has 0's in columns 7-11 our theorem exists. Otherwise it does not.

Running this algorithm on the matrix shape shown yields 33 which define theorems determining the angle between lines 12 and 13.

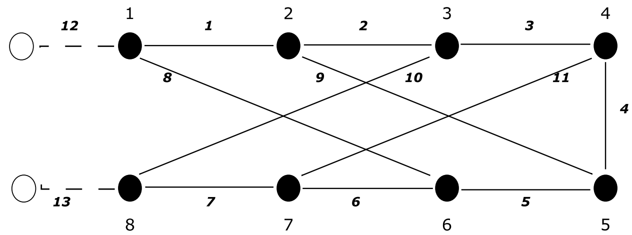

Figure 3: Graph representation of the matrix structure for Figure 2

4 Generalized Theorem Structure

We have taken a matrix which is known to generate a worthwhile theorem, and interpreted it in a different way geometrically. We have further taken a matrix whose shape is shared by two interesting theorems, varied the numeric values attributed to the non-zero entries and generated a collection of theorems sharing the matrix shape. We now would like to construct other matrices which have a good chance of generating interesting theorems. We narrow our consideration to matrices formed solely by the constraints 2, 3 and 4 above. Hence each hypothesis contributes a row to the matrix formulation of the problem comprising three entries (two -1's and a 2). The matrices derived from the theorems of figure 2 also have the following characteristics:

- All but two columns of the matrix contain 2 non-zero elements.
- Two rows both have non-zero entries in at most a single column.

Such a theorem can be represented as a graph in the following way (figure 3). Each vertex of the graph represents a row of the matrix. An edge joins two vertices if there are two non-zero entries in a column. If a column contains a single non-zero entry, then a node is added and an edge joining it to the node representing the row of the non-zero entry. In the figure, the numbers on the graph vertex correspond to row numbers in a matrix, whereas the numbers on the graph edges correspond to column numbers. This graph represents the matrix for the diagrams of figure 2. The two dashed lines attached to the 1st and 8th nodes correspond to the matrix columns with single non-zero entries, and thus to geometric lines whose angle would be determined. These nodes (unfilled in the diagram) do not correspond to matrix rows.

An associated cubic graph (uniform degree 3) may be defined by removing the white nodes and dashed edges and adding an edge between the two vertices with degree 2. Equivalently, in the matrix representation, we merge the two columns which have a single non-zero entry.

Reversing this process, given a cubic hamiltonian graph H, we create a graph G by removing one edge. Let u and v be the graph nodes adjacent to this edge. These have degree 2 in G. We number the nodes of G from 1 to $m = 2p$ and its edges from 1 to $n = 3p - 1$. We construct a matrix P such that for $j \leq n$ $P_{ij} = 1$ if vertex i is adjacent to edge j and 0 otherwise. We add two colums defined as follows:

$$P_{i,n+1} = \begin{cases} 1 & i = u \\ 0 & otherwise \end{cases}$$

$$P_{i,n+2} = \begin{cases} 1 & i = v \\ 0 & otherwise \end{cases}$$

A potential theorem matrix M is formed by assigning values to the non zero entries of the pattern matrix P. Given such a pattern with m rows, there are clearly 3^m different such assignments.

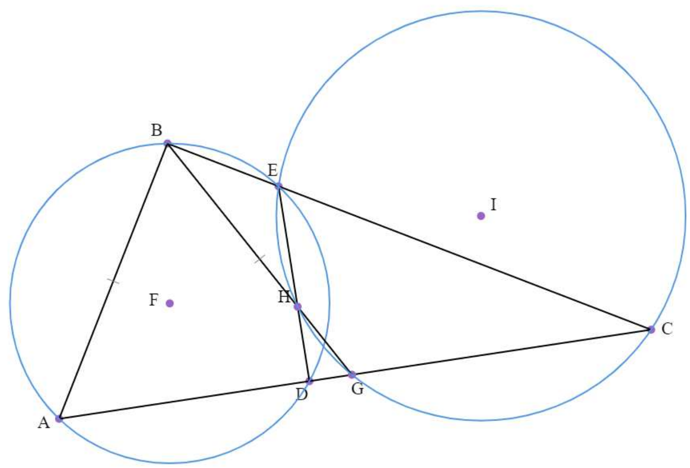

Figure 4: A Theorem Corresponding to a generated matrix. ABC is a triangle E and D lie on sides BC and AC such that $ABED$ is cyclic. G is on AC such that $|AB| = |BG|$. H is the intersection of BG and DE. Then C,E,H,G are cyclic.

When converting a matrix to geometry, there is a choice for each row of the matrix whether to interpret the row as an angle bisector (equivalently a reflection) or as an isosceles triangle. If two isosceles triangles share one of the equal sides, this can be represented geometrically as two chords of a circle meeting at a point. The line from the circle's center to the common point may be omitted from the diagram, and is implied by the geometry. If there are several opportunities to use the same circle, this can make the diagram and hence the theorem, much more appealing. In particular, we form the graph G' whose vertices correspond to the columns of the matrix, and whose edges correspond to the rows: each edge joins the two columns in that row whose values are -1. Cycles in G' can be made to correspond to cyclic polygons in the geometry figure and much clutter disappears.

As an example, a theorem generated from the 10 node pattern is depicted in figure 4.

5 Conclusion

Our approach to theorem discovery is to identify matrix patterns which, with appropriate numeric values lead to theorems. In this paper, we have narrowed our definition of what constitutes a 'theorem' to be that an angle between a pair of lines is determined by the hypotheses. We have also narrowed our focus to theorems which are described solely in terms of angle bisector (equivalently isosceles triangle) constraints.

The rows of our matrices, corresponding to bisector constraints, contain three non-zero entries, and their values are 2, -1, -1. We further specialize by specifying that all but one or two columns must contain exactly two non zero elements, and that column or columns should contain one. The locations of the non zero elements in matrices of this kind can be represented as a graph where each row corresponds to a graph node, and each column with two non-zero elements corresponds to a graph edge between the corresponding nodes.

A straightforward mechanistic approach to converting the matrix to a geometry theorem would be to consider every constraint to be simply an angle bisector. With this approach, the matrix of figure 1 yields the following theorem.

Given 11 lines A, B, C, D, E, F, G, H, I, J, K such that K is the angle bisector of A and G, H is the angle bisector of A and B and of D and E, C is the angle bisector of B and I and of D and J, and F is the angle bisector of E and J and of G and I, then lines E and K are parallel.

While straightforward to mechanize, the resulting theorem is far from elegant. Judicious choice of geometric representation for each matrix row, between isosceles triangle constraint, angle bisector, circle chord and reflection can greatly enhance the attractiveness of the theorem, as can considerations of symmetry. The development of a set of heuristics to drive the automation of this choice is a topic for further work.

A potential use of this capability, once developed would be for the automated generation of non-trivial proof problems for students. If a problem comes from a specific graph shape, its level of complexity could be controlled.

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