

Realizations of Rigid Graphs*

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A minimally rigid graph, also called Laman graph, models a planar framework which is rigid for a general choice of distances between its vertices. In other words, there are finitely many ways, up to isometries, to realize such a graph in the plane. Using ideas from algebraic and tropical geometry, we derive a recursive formula for the number of such realizations. Combining computational results with the construction of new rigid graphs via gluing techniques, we can give a new lower bound on the maximal possible number of realizations for graphs with a given number of vertices.

1 Introduction

The theory of rigid graphs forms a fascinating research area in the intersection of graph theory, computational algebraic geometry, and algorithms. The study of rigid structures, also called frameworks, was originally motivated by mechanics, and it goes back at least to the 19th century. Besides being a very interesting mathematical subject, rigid graphs and the underlying theory of Euclidean distance geometry have meanwhile found a large number of applications ranging from robotics and bioinformatics to sensor network localization and architecture.

Suppose that we are given a graph G with edge set E . We consider the set of all possible realizations (embeddings) of the graph in the Euclidean plane such that the lengths of the edges coincide with some prescribed edge labeling $\lambda : E \rightarrow \mathbb{R}_{>0}$. Edges and vertices are allowed to overlap in such a realization. For example, suppose that G has four vertices and is a complete graph minus one edge. Figure 1 shows all possible realizations of G up to rotations and translations, for a certain edge labeling. We address the following problem:

For a given graph determine its number of realizations for a general edge labeling, up to rotations and translations.

Here we say that a property holds for a *general edge labeling* if it holds for all edge labelings belonging to a dense open subset of the vector space of all edge labelings.

The realizations of a graph can be considered as physical structures in the plane, which consist of rods that are connected by rotational joints. If a graph together with an edge labeling admits infinitely (resp. finitely) many realizations up to rotations and translations, then the corresponding planar structure is flexible (resp. rigid), see Figure 2.

A graph is called *generically rigid* (or isostatic) if a general edge labeling yields a rigid realization. No edge in a generically rigid graph can be removed without losing rigidity, that is why such graphs are also called *minimally rigid* in the literature. The complete graph on four vertices K_4 is for instance not considered to be minimally generically rigid, since for a general choice of edge lengths it will not have a realization: imagine you would have to add the missing edge in either of the realizations depicted

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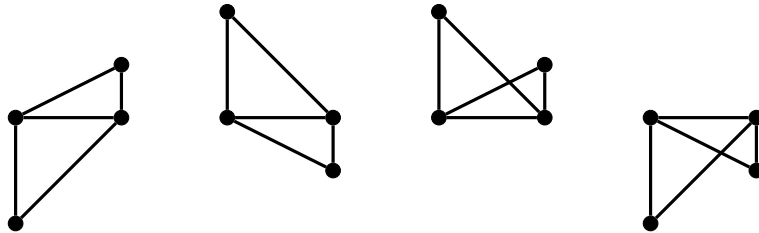


Figure 1: All four realizations of the minimally rigid graph with four vertices.

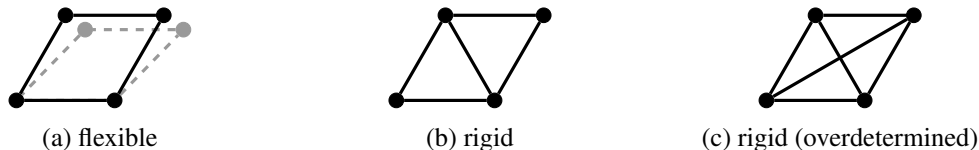


Figure 2: Graphs and their state of rigidity

in Figure 1 — for most prescribed lengths of the new edge this will not be possible, see also Figure 2. Hilda Pollaczek-Geiringer [9] and independently Gerard Laman [8] characterized the property of generic rigidity in terms of the number of edges and vertices of the graph and its subgraphs, hence such objects are also known as *Laman graphs*.

Theorem 1 *A graph $G = (V, E)$ is minimally (generically) rigid if and only if $|E| = 2|V| - 3$, and for every subgraph $G' = (V', E')$ with at least two vertices it holds $|E'| \leq 2|V'| - 3$.*

All finitely many realizations of a Laman graph can be obtained as the solution set of a system of quadratic polynomial equations, where the edge labels are either given by concrete numbers or interpreted as parameters. In general it is difficult to produce results on the number of real solutions of such systems. In such situations, one often switches to a complex setting; this also enables us to apply results from algebraic geometry. Hence, from now on, we consider edge labelings with complex numbers, and we are interested in the number of *complex solutions*, up to an equivalence relation on \mathbb{C}^2 generalizing the direct isometries of \mathbb{R}^2 ; this number is the same for any general edge labeling, so we call it the *Laman number* of the graph G , denoted by $\text{Lam}(G)$. For some graphs up to 8 vertices, this number had been computed using random values for the edge labels [7] — this means that it is very likely, but not absolutely certain, that these computations give the true numbers. Upper and lower bounds for Laman numbers are considered in [4, 10, 1]. Note that for many Laman graphs there exists a (real) edge labeling such that the number of real realizations equals precisely the Laman number. However, there are graphs for which the Laman number gives only an upper bound on the number of real realizations.

Our main result is a combinatorial algorithm that computes the number of complex realizations of any given Laman graph; it is much more efficient than just solving the corresponding nonlinear system of equations. The algorithm and its correctness proof are presented in detail in [2]. Using a supercomputer, we apply this algorithm to a large collection of Laman graphs and identify among all Laman graphs with n vertices ($n \leq 12$) the one with the maximal Laman number. This allows us to derive better lower bounds on the number of realizations [5]. In the following, we provide a concise summary of these results, focusing on the main ideas and the algorithmic point of view.

2 Computing the Laman Number

We write $G = (V, E)$ to denote a finite graph G with vertices V and edges E . An edge e between two vertices u and v is denoted by $\{u, v\}$; this notation expresses the fact that all graphs considered here are *undirected*.

Using nonnegative real labels for the edge lengths, the number of realizations in \mathbb{R}^2 for a general edge labeling is not well-defined, since it heavily depends on the actual labeling and not only on the graph. For example, the complete graph K_3 permits two different realizations (one being the reflection of the other) for almost all edge labelings that satisfy the triangle inequality, while it has none for all other labelings. In order to define a number that depends only on the graph, we switch to a complex setting. In order to keep notations simple, we take the convention that the edge labelings give the squared distances between vertices.

Definition 1 Let $G = (V, E)$ be a graph.

- ▷ A labeling of G is a function $\lambda: E \rightarrow \mathbb{C}$. The pair (G, λ) is called a labeled graph.
- ▷ A realization of G is a function $\rho: V \rightarrow \mathbb{C}^2$. Let λ be a labeling of G : we say that a realization ρ is compatible with λ if for each $e \in E$ the distance between its endpoints agrees with its label:

$$\lambda(e) = \langle \rho(u) - \rho(v), \rho(u) - \rho(v) \rangle, \quad e = \{u, v\},$$

where $\langle x, y \rangle = x_1y_1 + x_2y_2$.

A labeled graph (G, λ) is realizable if and only if there exists a realization ρ that is compatible with the edge labeling λ .

We say that two realizations of a graph G are equivalent if and only if there exists a direct isometry σ of \mathbb{C}^2 between them, where σ is a map of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + b,$$

where $A \in \mathbb{C}^{2 \times 2}$ is an orthogonal matrix with determinant 1 and $b \in \mathbb{C}^2$.

Definition 2 A labeled graph (G, λ) is called rigid if it is realizable and there are only finitely many realizations compatible with λ , up to equivalence.

Our main interest is to count the number of realizations of generically rigid graphs, namely graphs for which almost all realizable labelings induce rigidity.

Definition 3 A graph G is called generically realizable if for a general labeling λ the labeled graph (G, λ) is realizable. A graph G is called generically rigid if for a general labeling λ the labeled graph (G, λ) is rigid.

The number of realizations can be found by solving the following system of equations

$$\left((x_u - x_v)^2 + (y_u - y_v)^2 = \lambda_{uv} \right)_{\{u,v\} \in E}.$$

Equivalently, we can study the map r_G whose preimages of $(\lambda_{uv})_{\{u,v\} \in E}$ correspond to the solutions of the above system

$$r_G: \mathbb{C}^{2|V|} \rightarrow \mathbb{C}^E, \quad (x_v, y_v)_{v \in V} \mapsto \left((x_u - x_v)^2 + (y_u - y_v)^2 \right)_{\{u,v\} \in E}.$$

Still we get infinitely many solutions due to translations and rotations. Translations can be eliminated by moving one vertex to the origin. In order to handle rotations we perform the following transformation $x_v + iy_v \rightarrow x_v$, $x_v - iy_v \rightarrow y_v$. Then the above equations become

$$\left((x_u - x_v)(y_u - y_v) = \lambda_{uv} \right)_{\{u,v\} \in E}.$$

In this way, solutions that differ only by a rotation are the same in a suitable projective setting. If we transform the map r_G accordingly we obtain a map whose degree is finite and gives the sought number of realizations.

In order to set up a recursive formula for the degree of the map, we want to be able to handle the two factors $(x_u - x_v)$ and $(y_u - y_v)$ independently. To do this we duplicate the graph, and for technical reasons we allow a more general class of graphs. The resulting concept is roughly speaking a pair of graphs (G, H) with a bijection between their sets of edges. We identify edges by this bijection.

Definition 4 A bigraph is a pair of undirected graphs (G, H) — allowing several components, multiple edges and self-loops — where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. The set \mathcal{E} is called the set of biedges, and there are two maps that assign to each $e \in \mathcal{E}$ the corresponding vertices in V and W , respectively. Note that G and H are in general different graphs but there is a bijection between their sets of edges.

We define the Laman number $\text{Lam}(B)$ of a bigraph B as the degree of an associated map defined in a similar way as r_G . Moreover, we show that the Laman number of a graph equals the Laman number of the corresponding bigraph.

Proposition 1 The number of realizations $\text{Lam}(G)$ of a Laman graph G is equal to the Laman number $\text{Lam}(B)$ of the bigraph $B = (G, G)$.

The idea for proving the recursion formula in Theorem 2 is inspired by tropical geometry: we consider the equation system over the field of Puiseux series; an algebraic relation between Puiseux series implies a piecewise linear relation between their orders. We encode these piecewise linear relations in a combinatorial data which we call *bidistance*. A bidistance is a pair of functions from the edges of a bigraph to the rational numbers \mathbb{Q} , which satisfies certain conditions. Using a bidistance d of a bigraph B we can define a new bigraph B_d with the same number of edges. The solutions of the equations for the bigraph B that correspond to the bidistance d are in bijection with the solutions of the equations for B_d . Then the solutions for B are partitioned by the bidistances, implying the following formula for the Laman number:

$$\text{Lam}(B) = \sum_d \text{Lam}(B_d).$$

From this we finally show the combinatorial recursion formula. For doing so we prove that $\text{Lam}(B_d)$ is either easy to compute or the product of two Laman numbers of bigraphs with fewer edges each. We need some more notation to state the theorem.

Definition 5 Let $B = (G, H)$ be a bigraph with biedges \mathcal{E} , then we say that B is pseudo-Laman if $\dim(G) + \dim(H) = |\mathcal{E}| + 1$, where $\dim(G) := |V| - |\{\text{connected components of } G\}|$.

It can be easily seen, that if G is a Laman graph, then the bigraph (G, G) is pseudo-Laman. From a given bigraph we want to construct new ones with a smaller number of edges. We introduce two constructions, *quotient* and *complement*, both for usual graphs (see Figure 3) and for bigraphs.

Definition 6 Let $G = (V, E)$ be a graph, and let $E' \subseteq E$. We define two new graphs, denoted G / E' and $G \setminus E'$, as follows:

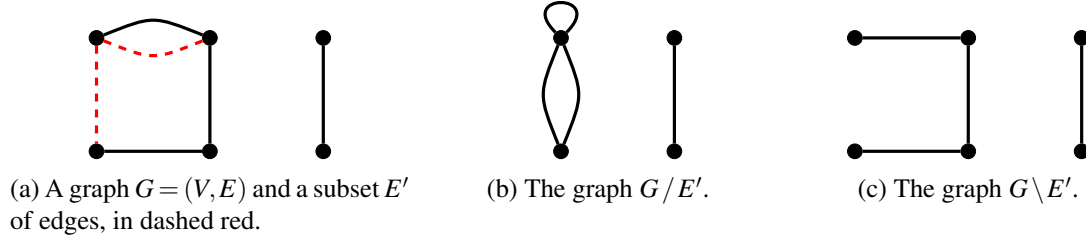


Figure 3: Example of the two constructions in Definition 6.

- ▷ Let G' be the subgraph of G determined by E' . Then we define G/E' to be the graph obtained as follows: its vertices are the equivalence classes of the vertices of G modulo the relation dictating that two vertices u and v are equivalent if there exists a path in G' connecting them; its edges are determined by edges in $E \setminus E'$.
- ▷ Let \hat{V} be the set of vertices of G that are endpoints of some edge not in E' . Set $\hat{E} = E \setminus E'$. Define $G \setminus E' = (\hat{V}, \hat{E})$.

Definition 7 Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Given $\mathcal{M} \subseteq \mathcal{E}$, we define two bigraphs ${}^{\mathcal{M}}B = (G/\mathcal{M}, H \setminus \mathcal{M})$ and $B^{\mathcal{M}} = (G \setminus \mathcal{M}, H/\mathcal{M})$, with the same set of bidedges $\mathcal{E}' = \mathcal{E} \setminus \mathcal{M}$.

Theorem 2 Let $B = (G, H)$ be a pseudo-Laman bigraph with bidedges \mathcal{E} . Let $\bar{e} \in \mathcal{E}$ be a fixed biedge, then

- ▷ If G or H has a self-loop, then $\text{Lam}(B) = 0$.
- ▷ If both G and H consist of a single edge joining two different vertices, then $\text{Lam}(B) = 1$.
- ▷ Otherwise

$$\text{Lam}(B) = \text{Lam}({}^{\{\bar{e}\}}B) + \text{Lam}(B^{\{\bar{e}\}}) + \sum_{(\mathcal{M}, \mathcal{N})} \text{Lam}({}^{\mathcal{M}}B) \cdot \text{Lam}(B^{\mathcal{N}}),$$

where each pair $(\mathcal{M}, \mathcal{N}) \subseteq \mathcal{E}^2$ satisfies $\mathcal{M} \cup \mathcal{N} = \mathcal{E}$ and $\mathcal{M} \cap \mathcal{N} = \{\bar{e}\}$, and where \mathcal{M} and \mathcal{N} are such that $|\mathcal{M}| \geq 2$, $|\mathcal{N}| \geq 2$, and both ${}^{\mathcal{M}}B$ and $B^{\mathcal{N}}$ are pseudo-Laman.

Although the algorithm resulting from Theorem 2 has exponential complexity, it is much faster than computing the Laman number from a parametrized system of polynomial equations, even if the parameters are substituted by random values. Using our recursion formula we were able to compute Laman numbers for all Laman graphs up to 13 vertices. Furthermore, we computed Laman numbers for single graphs up to 22 vertices, which was out of reach with the previous methods. Additional information including implementations in Mathematica and in C++ can be found at www.koutschan.de/data/laman/ and <https://zenodo.org/record/1245506>.

3 Bounds on the Number of Realizations

The first *upper bound* [1] on the number of realizations of rigid graphs was derived using degree bounds from algebraic geometry. Based on the theory of distance matrices and determinantal varieties, the upper bound $\binom{2n-4}{n-2} = \Theta(4^n/\sqrt{n})$ is obtained, where n denotes the number of vertices. This bound

was improved [10] by exploiting the sparsity of the underlying polynomial systems, and it was further improved by applying additional tricks to take advantage of the sparsity and the common sub-expressions that appear in the polynomial systems [4]. A direct application of mixed volume techniques, which capture the sparsity of a polynomial system, yields a bound of 4^{n-2} . If one also takes into account the degree of the vertices, then for a Laman graph with $k \geq 4$ degree-2 vertices, the number of realizations of G is bounded from above by $2^{k-4}4^{n-k}$.

The first *lower bounds* for the number of realizations of Laman graphs were $24^{\lfloor (n-2)/4 \rfloor}$ (approx. 2.21^n) and $2 \cdot 12^{\lfloor (n-3)/3 \rfloor}$ (approx. 2.29^n), which exploited a gluing process using a caterpillar, resp. fan construction [1]. Both constructions use the three-prism graph (sometimes also called Desargues graph) as a building block, which is a graph with $n = 6$ vertices and 24 realizations. More recent lower bounds are 2.30^n from [3] and 2.41^n from [7].

We derive better lower bounds on the maximal number of complex realizations of minimally rigid graphs with a prescribed number of vertices. Clearly, the number of complex realizations is an upper bound on the number of real realizations. It is known [7] that the numbers of real and complex realizations do not match in general, and it is an interesting problem to quantify this gap. On the one hand, one can construct infinite families of graphs for which the ratio between real and complex realizations tends to zero. On the other hand, there are graphs, see [3] for a nontrivial example, where real edge lengths can be found such that there exist as many real realizations as complex ones.

Using the novel algorithm presented in Section 2 we compute the exact number of realizations for graphs with a relatively small number of vertices. Then we introduce techniques to “glue” an arbitrary number of such small graphs in order to produce graphs with a high number of vertices (and edges) that preserve rigidity. The gluing process allows us to derive the number of realizations of the final graph from the number of realizations of its components, and in this way we derive a lower bound for the number of realizations in \mathbb{C}^2 . Moreover, we perform extensive experiments in order to identify those small graphs that attain the maximum number of realizations and that can be the building blocks for the gluing process.

Definition 8 *We define $M(n)$ to be the largest Laman number that is achieved among all Laman graphs with n vertices.*

3.1 Constructions

We discuss different constructions of infinite families of Laman graphs $(G_n)_{n \in \mathbb{N}}$ with G_n having n vertices. We do this in a way such that we know precisely the Laman number for each member of the family. This directly leads to a lower bound on $M(n)$. The ideas of these constructions are described in [1]; they were used to get lower bounds by connecting several three-prism graphs at a common basis. Here, we generalize them in order to connect any Laman graphs at an arbitrary Laman base. We present three such constructions.

The *caterpillar construction* [1] works as follows: place k copies of a Laman graph $G = (V, E)$ in a row and connect every two neighboring ones by means of a shared edge (see Figure 4). Alternatively, one can let all k graphs share the same edge. In any case, the resulting assembly has $2 + k(|V| - 2)$ vertices and its Laman number is $\text{Lam}(G)^k$, since each of the k copies of G can achieve all its $\text{Lam}(G)$ different realizations, independently of what happens with the other copies. Hence, among all Laman graphs with $n = 2 + k(|V| - 2)$ vertices there exists one with $\text{Lam}(G)^k$ realizations. If the number of vertices n is not of the form $2 + k(|V| - 2)$ then we can use the previous caterpillar graph with $\lfloor (n - 2) / (|V| - 2) \rfloor$ copies of G and perform some Henneberg steps of type 1: such a step adds one vertex and connects it to two

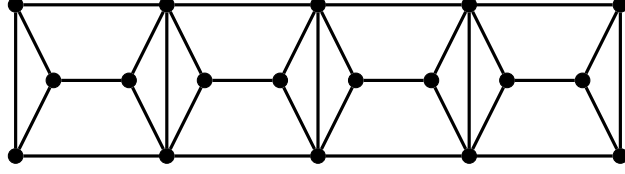


Figure 4: Caterpillar construction with 4 copies of the three-prism graph.

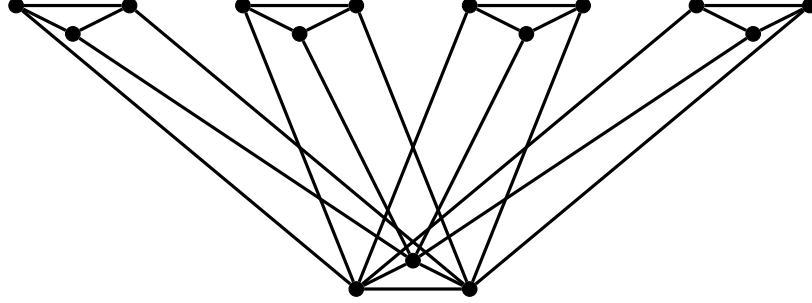


Figure 5: Fan construction with 4 copies of the three-prism graph.

existing vertices, thereby doubling the Laman number. Summarizing, for any Laman graph G , we obtain the following lower bound from the caterpillar construction:

$$M(n) \geq 2^{(n-2) \bmod (|V|-2)} \cdot \text{Lam}(G)^{\lfloor (n-2)/(|V|-2) \rfloor} \quad (n \geq 2).$$

The second construction is the *fan construction*: take a Laman graph $G = (V, E)$ that contains a triangle (i.e., a 3-cycle), and glue k copies of G along that triangle (see Figure 5). Once we fix one of the two possible realizations of that triangle, each copy of G admits $\text{Lam}(G)/2$ realizations. The remaining $\text{Lam}(G)/2$ realizations are obtained by mirroring, i.e., by using the second realization of the common triangle. Similarly as before, the assembled fan is a Laman graph with $3 + k(|V| - 3)$ vertices that admits $2 \cdot (\text{Lam}(G)/2)^k$ realizations. Hence, we get the following lower bound:

$$M(n) \geq 2^{(n-3) \bmod (|V|-3)} \cdot 2 \cdot \left(\frac{\text{Lam}(G)}{2} \right)^{\lfloor (n-3)/(|V|-3) \rfloor} \quad (n \geq 3).$$

While the caterpillar construction can be done with any Laman graph, this is not the case with the fan. For example, the Laman graph with 12 vertices displayed in Figure 7 has no 3-cycle and therefore cannot be used for the fan construction.

As a third construction, we propose the *generalized fan construction*: instead of a triangle, we may use any Laman subgraph $H = (W, F)$ of G for gluing. Using k copies of G , we end up with a fan consisting of $|W| + k(|V| - |W|)$ vertices and Laman number at least $\text{Lam}(H) \cdot (\text{Lam}(G)/\text{Lam}(H))^k$. Here we assume that the realizations of G are divided into $L(H)$ equivalence classes of equal size, by considering two realizations of G as equivalent if the induced realizations of H are equal (up to rotations and translations). If this assumption was violated, the resulting lower bound would be even better; thus we can safely state the following bound:

$$M(n) \geq 2^{(n-|W|) \bmod (|V|-|W|)} \cdot \text{Lam}(H) \cdot \left(\frac{\text{Lam}(G)}{\text{Lam}(H)} \right)^{\lfloor (n-|W|)/(|V|-|W|) \rfloor} \quad (n \geq |W|).$$

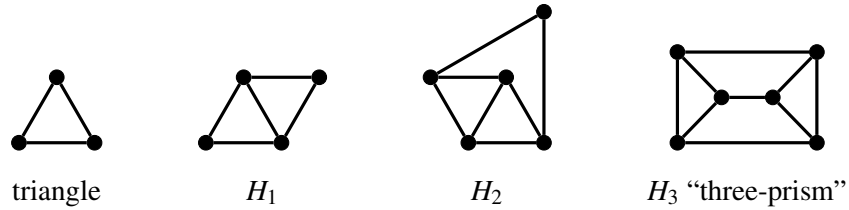


Figure 6: Bases for the generalized fan construction and their encodings.

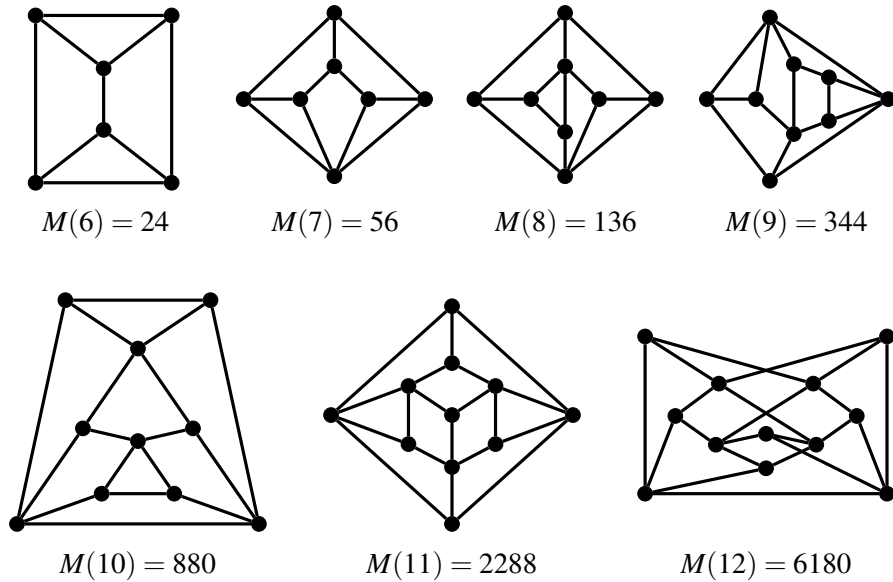


Figure 7: Unique Laman graphs with $6 \leq n \leq 12$ with maximal number of realizations

Note that the previously described fan construction is a special instance of the generalized fan, by taking as the subgraph H a triangle with $\text{Lam}(H) = 2$. To indicate the subgraph of a generalized fan construction we also write H -fan. The fan fixing the 4-vertex Laman graph is then denoted by H_1 -fan (see Figure 6 for these base graphs and their naming convention).

3.2 Lower bounds

In order to get good lower bounds, we need particular Laman graphs that have a large number of realizations. For this purpose we have computed the Laman numbers of all Laman graphs with up to $n = 13$ vertices. For each $3 \leq n \leq 12$ we have identified the (unique) Laman graph with the highest number of realizations. We present these numbers and the corresponding graphs for $6 \leq n \leq 12$ in Figure 7.

However, there are 44 176 717 Laman graphs with 12 vertices, and it took 56 processor days to compute the Laman numbers of all of them. Going through all 1 092 493 042 Laman graphs with 13 vertices was an even more challenging undertaking, and it is unrealistic to do the same for larger Laman graphs. In order to proceed further, we developed some heuristics to construct graphs with very high Laman numbers, albeit not necessarily the highest one.

We now use these results to derive new and better lower bounds than the previously known ones. We apply the caterpillar construction to the Laman graphs with the maximal number of realizations for $6 \leq$

n	caterpillar	fan	H_1 -fan	H_2 -fan	H_3 -fan
6	2.21336	2.28943	2	2	-
7	2.23685	2.30033	2.28943	2	2
8	2.26772	2.32542	2.30033	2.28943	2
9	2.30338	2.35824	2.35216	2.30033	2.28943
10	2.33378	2.38581	2.35824	2.35216	2.30033
11	2.36196	2.41159	2.38581	2.35824	2.35216
12	2.39386	2.43198	2.43006	2.39802	2.35824
13	2.40453	2.44498	2.44772	2.42197	2.39802
14	2.43185	2.46087	2.46391	2.44251	2.42197
15	2.44695	2.47445	2.47076	2.45031	2.42906
16	2.46890	2.48657	2.48794	2.47166	2.43712
17	2.48875	2.49779	2.49160	2.48043	2.46341
18	2.49378	2.50798			

Table 1: Growth rates (rounded) of the lower bounds. For $n \leq 13$ these values are proven to be the best achievable ones; for $n > 13$ the values are just the best we found by experiments, hence it is possible that there are better ones. The drawings of the graphs corresponding to the last three columns are given in Figure 6.

$n \leq 13$, and for $14 \leq n \leq 18$ we use graphs with high Laman numbers that were found heuristically. The fan construction is applied to the maximal Laman graphs for $6 \leq n \leq 11$ only, since it is not applicable to the maximal graph with 12 vertices, because that graph does not contain K_3 as a subgraph (see Figure 7). All lower bounds that we obtained by these constructions are summarized in Table 1.

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