Reversible Languages Having Finitely Many Reduced Automata

Kitti Gelle
University of Szeged
Szeged, Hungary
kgelle@inf.u-szeged.hu

Szabolcs Iván
University of Szeged
Szeged, Hungary*
szabivan@inf.u-szeged.hu

Reversible forms of computations are often interesting from an energy efficiency point of view. When the computation device in question is an automaton, it is known that the minimal reversible automaton recognizing a given language is not necessarily unique, moreover, there are languages having arbitrarily large reversible recognizers possessing no nontrivial reversible congruence. However, the exact characterization of this class of languages was open. In this paper we give a forbidden pattern capturing the reversible regular languages having only finitely many reduced reversible automata, allowing an efficient (NL) decision procedure.

1 Introduction

Landauer's principle [12] states that any logically irreversible manipulation of information – such as the merging of two computation paths – is accompanied by a corresponding entropy increase of non-information-bearing degrees of freedom in the processing apparatus. In practice, this can be read as "merging two computation paths generates heat", though it has been demonstrated [22] that the entropy cost can be taken in e.g. angular momentum. Being a principle in physics, there is some debate regarding its validity, challenged [5, 19, 20] and defended [4, 3, 11] several times recently. In any case, the study of *reversible* computations, in which distinct computation paths never merge, looks appealing. In the context of quantum computing, one allows only reversible logic gates [18]. For classical Turing machines, it is known that each deterministic machine can be simulated by a reversible one, using the same amount of space [2, 13]. Hence, each regular language is accepted by a reversible *two-way* deterministic finite automaton (also shown in [8]).

In the case of classical, i.e. *one-way* automata, the situation is different: not all regular languages can be recognized by a reversible automaton, not even if we allow partial automata (that is, trap states can be removed, thus the transition function being a partial one). Those languages that can be recognized by a reversible one are called *reversible languages*. It is clear that one has to allow being partially defined at least since otherwise exactly the regular group languages (those languages in whose minimal automata each letter induces a permutation) would be reversible.

Several variants of reversible automata were defined and studied [21, 1, 17]. The variant we work with (partial deterministic automata with a single initial state and and arbitrary set of final states) have been treated in [9, 10, 6, 16, 15, 14]. In particular, in [6] the class of reversible languages is characterized by means of a forbidden pattern in the minimal automaton of the language in question, and an algorithm is provided to compute a minimal reversible automaton for a reversible language, given its minimal automaton. Here "minimal" means minimizing the number of states, and the minimal reversible automaton is shown to be not necessarily unique. In [16], the notion of *reduced* reversible automata is introduced:

*Research supported by NKFI Grant no. 108448.

E. Csuhaj-Varjú, P. Dömösi, Gy. Vaszil (Eds.): 15th International Conference on Automata and Formal Languages (AFL 2017)
EPTCS 252, 2017, pp. 114–127, doi:10.4204/EPTCS.252.13

a reversible automaton is reduced if it is trim (all of its states are accessible and coaccessible), and none of its nontrivial factor automata is reversible. The authors characterize the class of those reversible languages (again, by developing a forbidden pattern) having a unique reduced reversible automaton (up to isomorphism), and leave open the problem to find a characterization of the class of those reversible languages having finitely many reduced reversible automata (up to isomorphism).

In this paper we solve this open problem of [16], by also developing a forbidden pattern characterization which allows an **NL** algorithm.

2 Notation

We assume the reader has some knowledge in automata and formal language theory (see e.g. [7]).

In this paper we consider deterministic *partial* automata with a single initial state and an arbitrary set of final states. That is, an *automaton*, or DFA, is a tuple $M=(Q,\Sigma,\delta,q_0,F)$ with Q being the finite *set of states*, $q_0 \in Q$ the *initial* state, $F \subseteq Q$ the set of *final* or *accepting* states, Σ the finite, nonempty *input alphabet* of symbols or *letters* and $\delta : Q\times\Sigma \to Q$ the *partial* transition function which is extended in the usual way to a partial function also denoted by $\delta : Q\times\Sigma^* \to Q$ with $\delta(q,\varepsilon)=q$ for the *empty word* ε and $\delta(q,wa)=\delta(\delta(q,w),a)$ if $\delta(q,w)$ is defined and undefined otherwise. When δ is understood from the context, we write $q\cdot w$ or qw for $\delta(q,w)$ in order to ease notation. When M is an automaton and q is a state of M, then *the language recognized by M from q* is $L(M,q)=\{w\in\Sigma^* : qw\in F\}$. The *language recognized by M* is $L(M)=L(M,q_0)$. A language is called *regular* or *recognizable* if some automaton recognizes it.

When p and q are states of the automata M and N, respectively, we say that p and q are *equivalent*, denoted $p\equiv q$, if $L(M,p)=L(N,q)$. (When M or N is unclear from the context, we may write $(M,p)\equiv(N,q)$.) The automata M and N are said to be equivalent if their initial states are equivalent.

A state q of M is *useful* if it is *reachable* ($q_0w=q$ for some w) and *productive* ($qw\in F$ for some w). A DFA is *trim* if it only has useful states. When $L(M)$ is nonempty, one can erase the non-useful states of M: the resulting automaton will be trim and equivalent to M (and may be partially defined even if M is totally defined, if M has a trap state q for which $L(M,q)=\emptyset$). An equivalence relation Θ on the state set Q is a *congruence* of M if $p\Theta q$ implies both $p\in F \Leftrightarrow q\in F$ (that is, F is *saturated* by Θ) and $pa\Theta qa$ for each $a\in\Sigma$ (that is, Θ is *compatible* with the action). In particular, in any Θ-class, pa is defined if and only if so is qa.

Clearly the identity relation Δ_Q on Q is always a congruence, the *trivial* congruence. A trim automaton is *reduced* if it has no nontrivial congruence. When Θ is a congruence of M and $p\ \Theta\ q$ are states falling into the same Θ-class, then $p\equiv q$. Given M and a congruence Θ on M, the *factor automaton* of M is $M/\Theta = (Q/\Theta,\Sigma,\delta/\Theta,q_0/\Theta,F/\Theta)$ where p/Θ denotes the Θ-class of p, X/Θ denotes the set $\{p/\Theta :\ p\in X\}$ of Θ-classes for a set $X\subseteq Q$ and $\delta(q/\Theta,a)=\delta(q,a)/\Theta$ if $\delta(q,a)$ is defined, and is undefined otherwise.

Then, for each $p\in Q$ the states p and p/Θ are equivalent, thus any automaton is equivalent to each of its factor automata. It is also known that for any automaton M there is a unique (up to isomorphism, i.e. modulo renaming states) equivalent reduced automaton, the one we get by trimming M, then factoring the useful part of M by the language equivalence relation $p\Theta_M q \Leftrightarrow p\equiv q$.

Any automaton can be seen as an edge-labeled multigraph and thus its *strongly connected components*, or SCCs, are well-defined classes of its states: the states p and q belong to the same SCC if $pu=q$ and $qv=p$ for some words $u,v\in\Sigma^*$. Clearly, this is an equivalence relation. We call an SCC *trivial* if it consists of a single state p with $pu\neq p$ for any nonempty word u (that is, if it contains absolutely no edges, not even loops), and *nontrivial* otherwise.

3 Reversible languages

An automaton $M = (Q, \Sigma, \delta, q_0, F)$ is *reversible* if $pa = qa$ implies $p = q$ for each $p, q \in Q$ and $a \in \Sigma$. A language $L \subseteq \Sigma^*$ is *reversible* if it is recognizable by some reversible automaton. A *reversible congruence* of a reversible automaton M is a congruence Θ of M such that the factor automaton M/Θ is also reversible. The automaton M is a *reduced reversible automaton* if it has no nontrivial reversible congruence.

It is known [6] that a language is reversible if and only if its minimal automaton has no distinct states $p \neq q$, a letter a and a word w such that $pa = qa$ and $paw = w$ (the forbidden pattern is depicted on Figure 1). Equivalently, for any state r belonging to a nontrivial component, and letter a, the set $\{p \in Q : pa = r\}$ has to have at most one element.

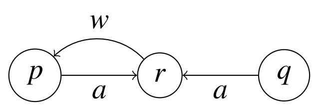
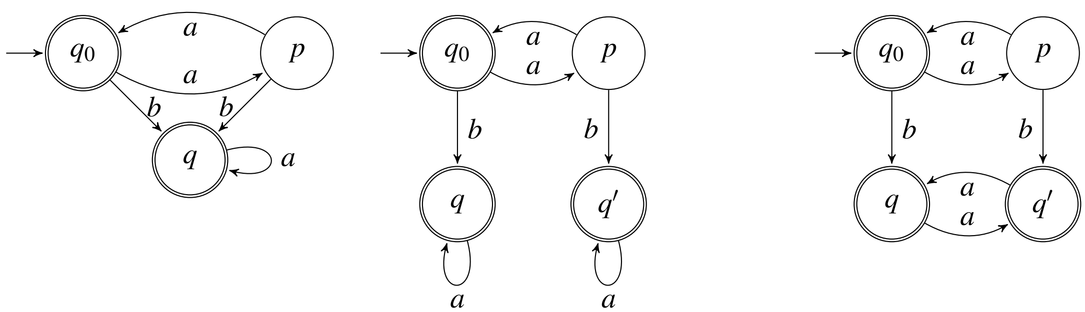


Figure 1: The forbidden pattern for reversible languages

Contrary to the general case of regular languages, there can be more than one reduced reversible automata, up to isomorphism, recognizing the same (reversible) language. For example, see Figure 2 of [16].



(a) The minimal DFA (b) A reduced reversible DFA (c) Another reduced reversible DFA

Figure 2: The case of the language $(aa)^* + a^*ba^*$

In that example the minimal automaton (depicted on Subfigure (a)) is not reversible and there are two nonisomorphic reversible reduced automata recognizing the same language (with four states). We note that in this particular example there are actually an infinite number of reduced reversible automata, recognizing the same language. In [16] the set of states of a minimal automaton was partitioned into two classes: the *irreversible* states are such states which are reachable from some distinct states $p \neq q$ with the same word w, while the *reversible* states are those which are not irreversible. For example, in the case of Figure 2, q is the only irreversible state. One of the results of [16] is that if there exists an irreversible state p which is reachable from a nontrivial SCC of the automaton (allowing the case when p itself belongs to a nontrivial SCC, as q does in the the example), then there exist an infinite number of nonisomorphic reduced reversible automata, each recognizing the same (reversible) language. A natural question is to precisely characterize the class of these reversible languages.

4 Result

In this section we give a forbidden pattern characterization for those reversible languages having a finite number of reduced reversible automata, up to isomorphism.

For this part, let us fix a reversible language L. Let $M = (Q^*, \Sigma, \delta^*, q_0^*, F^*)$ be the minimal automaton of L. We partition the states of M into classes as follows: a state q is a...

- 1-*state* if there exists exactly one word u with $q_0^* u = q$;
- ∞-*state* if there exist infinitely many words u with $q_0^* u = q$;
- $\oplus$-*state* if it is neither a 1-state nor an ∞-state

and orthogonally, q is an...

- *irreversible state* if there are distinct states $p_1^* \neq p_2^* \in Q^*$ and a word u such that $p_1^* u = p_2^* u = q$;
- *reversible* if it is not irreversible.

As an example, consider Figure 3.

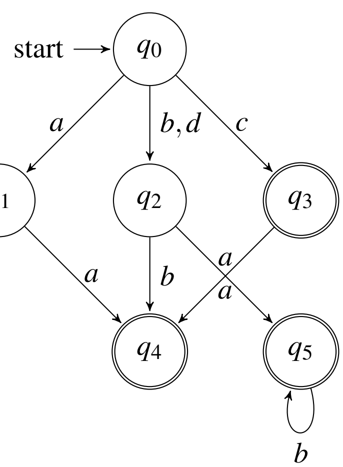


Figure 3: The minimal automaton of our running example language

Here, states q_0, q_1 and q_3 are 1-states, reachable by the words ε, a and c, respectively; q_2 and q_4 are $\oplus$-states as they are reachable by $\{b,d\}$ and $\{aa,bb,db,ca\}$, respectively and q_5 is a ∞-state, reachable by words of the form $(b+d)ab^*$. Moreover, q_4 is the only irreversible state (as $q_1 a = q_3 a$).

We note that our notion of irreversible states is not exactly the same as in [16]: what we call irreversible states are those states which belong to the "irreversible part" of the automaton in the terms of [16]. There, a state q is called irreversible only if $p_1 a = p_2 a = q$ for some distinct pair $p_1 \neq p_2$ of states and *letter* a.

Clearly, a state is an ∞-state iff it can be reached from some nontrivial SCC of M. Now we define the set $Z \subseteq Q^*$ of *zig-zag states*[1] as follows: Z is the least set X satisfying the following conditions:

1. All the ∞-states belong to X.
2. If $q \in X$ and $a \in \Sigma$ is a letter with $q \cdot a$ being defined, then $q \cdot a \in X$.
3. If q is a $\oplus$-state and $a \in \Sigma$ is a letter with $q \cdot a \in X$, then $q \in X$.

The main result of the paper is the following:

Theorem 1. *There are only finitely many reduced reversible automata recognizing L if and only if every zig-zag state of M is reversible.*

We break the proof up into several parts.

4.1 When all the zig-zag states are reversible

In this part we show that whenever all the zig-zag states are reversible, and N is a trim reversible automaton recognizing L, then there is a reversible congruence Θ on N such that N/Θ has a bounded number of states (the bound in question is computable from the minimal automaton M). So let us assume that there is no irreversible zig-zag state in M and let $N = (Q, \Sigma, \delta, q_0, F)$ be a trim reversible automaton recognizing L. Then, for each $q \in Q$ there exists a unique state $q^* \in Q^*$ with $q \equiv q^*$, and the function $q \mapsto q^*$ is a homomorphism.

Now let us define the relation Θ on Q as follows:

$$p \Theta q \quad \Leftrightarrow \quad (p = q) \ \text{or} \ (p^* = q^* \in Z).$$

Lemma 1. *The relation Θ is a reversible congruence on N.*

Proof. It is clear that Θ is an equivalence relation: reflexivity and symmetry are trivial, and $p \ \Theta \ q \ \Theta \ r$ either entails $p = q$ or $q = r$ (in which case $p \ \Theta \ r$ is obvious) or that $p^* = q^* = r^* \in Z$ (and then, $p \ \Theta \ r$ also holds).

Now if $p \ \Theta \ q$ and $p \cdot a$ is defined, then we have to show that $q \cdot a$ is also defined and $p \cdot a \ \Theta \ q \cdot a$. This is clear if $p = q$. Otherwise we have $p^* = q^*$ is a zig-zag state, thus $(p \cdot a)^* = (q \cdot a)^*$ as starring is a homomorphism from N to M (thus in particular, $q \cdot a$ is defined). As $p^* \in Z$ and Z is closed under action, we have that $p^* \cdot a = (p \cdot a)^*$ is also a zig-zag state, thus $p \cdot a \ \Theta \ q \cdot a$ indeed holds and Θ is a congruence on N.

To see that Θ is a reversible congruence, assume $p \cdot a \ \Theta \ q \cdot a$. We have to show that $p \ \Theta \ q$. If $p \cdot a = q \cdot a$, then $p = q$ (thus $p \ \Theta \ q$), since N is a reversible automaton. Otherwise, let $p \cdot a \neq q \cdot a$ (hence $p \neq q$) and $(p \cdot a)^* = (q \cdot a)^* \in Z$. By assumption on M, this state $(p \cdot a)^*$ is reversible. Thus, as $p^* \cdot a = q^* \cdot a = (p \cdot a)^*$, we get that $p^* = q^*$. Hence to show $p^* \ \Theta \ q^*$ it suffices to show that it is also a zig-zag state. If p^* is a ∞-state, then it is a zig-zag state by definition of Z. Also, if p^* is a $\oplus$-state, then it is still a zig-zag state (as $p^* \cdot a$ is a zig-zag state, we can apply Condition 3 in the definition of Z). Finally, observe that p^* cannot be a 1-state since p and q are different (reachable) states of N, hence there are words $u \neq v$ with $q_0 u = p$ and $q_0 v = q$. For these words, by starring being a homomorphism we get that $q_0^* u = q_0^* v = p^* (= q^*)$, thus p^* is reachable by at least two distinct words.

Hence, Θ is indeed a reversible congruence. $\square$

[1] The coined term "zig-zag" originates from an earlier version of Figure 5 on which forward edges had a "northeast" direction while backward edges had a "southeast" direction.

To conclude this case observe the following facts:

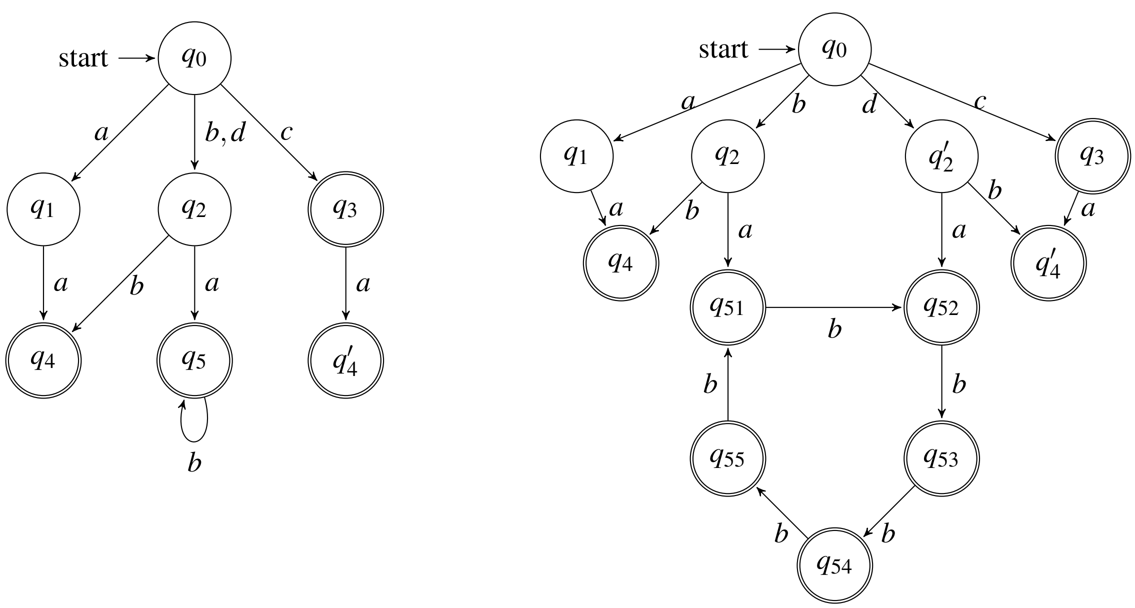
- For each non-∞-state p^* there exist a *finite* number $u_1, \ldots, u_{n(p^*)}$ of words leading into p^* in M, thus there can be at most $n(p^*)$ states in N which are equivalent to p^* (since N is trim). This bound $n(p^*)$ is computable from M.
- If p^* is a ∞-state of M, then p^* is a zig-zag state, thus all the states of N equivalent to p^* are collapsed into a single class of Θ. For these states, let us define the value $n(p^*)$ to be 1.

Hence, $n = \sum_{p^* \in Q^*} n(p^*)$ is a (finite, computable) upper bound on the number of states in the factor automaton N/Θ (hence it is an upper bound for the number of states in any reduced reversible automaton recognizing L as in that case Θ has to be the trivial congruence). Thus we have proved the first part of Theorem 1:

Theorem 2. *If all the zig-zag states of M are reversible, then there is a finite upper bound for the number of states of any reduced reversible automata recognizing L. Hence, in that case there exists only a finite number of nonisomorphic reduced reversible automata recognizing L.*

4.2 When there is an irreversible zig-zag state in M

In this part let us assume that M has an irreversible zig-zag state. We will start from an arbitrary reduced reversible automaton N recognizing L, and then "blow it up" to some arbitrarily large equivalent reduced reversible automaton. Before giving the construction, we illustrate the process in the case of Figure 3 (there, q_4 is an irreversible zig-zag state).



(a) Reversible automaton N equivalent to M

(b) The state q_5 is blown up by 5, yielding N'

Figure 4: Blowing up a reversible automaton

On Figure 4 (a), we have a reversible automaton N. On (b), we replace the state q_5 having a loop by a cycle of length 5 (and we also duplicate the state q_2 – we can do that since q_2 is not a 1-state). Then, the automaton N' is a reduced reversible automaton, whenever the length of the cycle (which is now chosen to be 5) is a prime number [16].

Indeed, any reversible congruence Θ on N' collapses equivalent states only. Suppose e.g. Θ collapses q_{51} and q_{53}. Applying b we get that q_{52} and q_{54} also get collapsed, and so q_{53} and q_{55}, and q_{54} and q_{51} – thus, all the q_{5x} states fall into a single Θ-class then. As Θ is assumed to be reversible, it has to collapse q_2 and q_2' as well; applying b also the states q_4 and q_4' have to be collapsed and finally, applying again reversibility we get that q_1 and q_3 should be collapsed but this cannot happen as they are not equivalent states. This reasoning works for any choice of different copies of q_5 (as far as the number chosen is a prime), thus N' has only the trivial reversible congruence and is a reduced reversible automaton.

The careful reader might realize that we actually followed in this reasoning a zig-zag path from the ∞-state q_5 to the irreversible state q_4 during the above reasoning. In this part we show that this approach can always be generalized whenever there exists an irreversible zig-zag state in M.

By the definition of the zig-zag states, if there exists some irreversible zig-zag state, then there is a sequence

$$r_0,\ (a_1,e_1),\ r_1,\ (a_2,e_2),\ \ldots,\ (a_\ell,e_\ell),\ r_\ell$$

such that $a_i \in \Sigma$ and $e_i \in \{+,-\}$ for each $1 \le i \le \ell$, and $r_i \in Q^*$ are states of M for each $0 \le i \le \ell$, moreover,

(i) if $e_i = +$ then $r_{i-1} \cdot a_i = r_i$ (denoted by $r_{i-1} \xrightarrow{a_i} r_i$ in the examples and patterns),

(ii) if $e_i = -$ then $r_i \cdot a_i = r_{i-1}$ and r_i is a $\oplus$-state (denoted by $r_{i-1} \xleftarrow{a_i} r_i$),

(iii) r_ℓ is an irreversible state,

(iv) r_0 is an ∞-state.

Let us choose such a sequence of minimal length. Then, by minimality,

- the states r_i are pairwise different,
- all the states r_i, $1 \le i < \ell$ are reversible $\oplus$-states.

To see that all the states are $\oplus$-states (them being pairwise different reversible states is obvious), observe that if r_i is an ∞-state for $0 < i$, then $r_i, \ldots, r_\ell$ is a shorter such sequence. Hence all the states r_i, $0 < i$ are either 1-states or $\oplus$-states. We show by induction that all of them are $\oplus$-states. The claim holds for $i = 1$ as $e_1 = +$ would imply that r_1 should be an ∞-state which cannot happen thus $e_1 = -$, hence r_1 is an $\oplus$-state. Now if r_i is an $\oplus$-state, then either $r_{i+1} = r_i a_{i+1}$ (if $e_{i+1} = +$) which implies that r_{i+1} cannot be a 1-state (thus it is an $\oplus$-state), or r_{i+1} is an $\oplus$-state (if $e_{i+1} = -$, applying ii)).

Now we extend the above sequence in both directions as follows.

First, r_0 being an ∞-state implies that there exists a state p_0 belonging to a nontrivial SCC of M (that is, $p_0 w = p_0$ for some nonempty word w) from which r_0 is reachable. That is, there is a word $b_1 b_2 \ldots b_m$ and states $p_1, \ldots, p_m$ with $p_m = r_0$ and $p_i b_{i+1} = p_{i+1}$ for each $0 \le i < m$.

Second, r_ℓ being an irreversible state implies that there exist different states s and s' of M from which r_ℓ is reachable by the same (nonempty) word. That is, there is a word $c_1 c_2 \ldots c_n$ and states $s_1, s_2, \ldots, s_n = r_\ell$ such that $sc_1 = s'c_1 = s_1$ and $s_i c_{i+1} = s_{i+1}$ for each $1 \le i < n$. See Figure 5.

Note that all the states $s_1, \ldots, s_n$ are $\oplus$-states and $p_0, \ldots, p_m$ are ∞-states.

In order to reduce the clutter in the notation, we treat the whole sequence $p_0, \ldots, p_{m-1}, r_0, \ldots, s_1, s$ and s' as a single indexed sequence $p_0, \ldots, p_{t-1}, p_{t,1}$ and $p_{t,2}$ (see Figure 6). Observe that each p_i (but possibly $p_{t,1}$ and $p_{t,2}$) is either an ∞- or a $\oplus$-state, and each of them is a zig-zag state. Moreover, all these states are pairwise different (thus inequivalent).

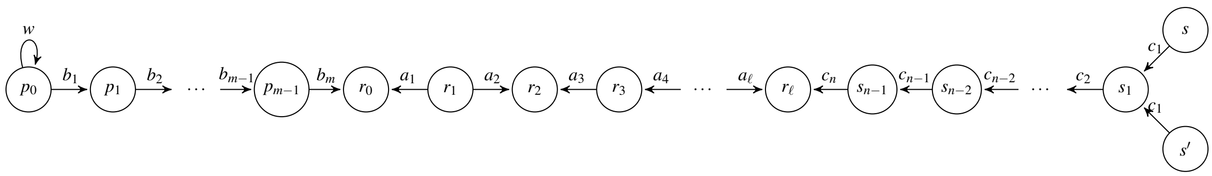


Figure 5: The sequences p_i, r_i and s_i.

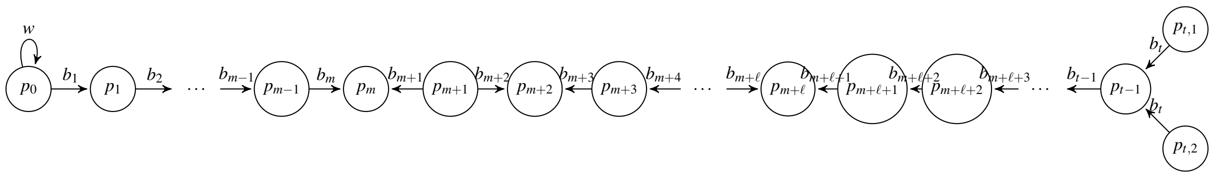


Figure 6: The sequence appearing in M, in an uniform notation

In the first step we show that if there is a specific pattern (which is a bit more general than a cycle of prime length) appears in a reversible automaton N', then every factor automaton of N' is "large".

Lemma 2. *Assume N' is a reversible automaton, $k \geq 1$ is a prime number, $t \geq 0$ and $i < k$ are integers, $q_0, \ldots, q_{k-1}$, $p'_1, \ldots, p'_t$, $p''_1, \ldots, p''_t$ are states, $a_1, \ldots, a_t \in \Sigma$ are letters, $e_1, \ldots, e_t \in \{+, -\}$ are directions and $w \in \Sigma^+$ is a word satisfying the following conditions:*

- *p'_t is not equivalent to p''_t,*
- *$q_j w = q_{j+1}$ for each $0 \leq j < k$ with the convention that $q_k = q_0$, that is, indices of the qs are taken modulo k,*
- *for each $1 \leq j \leq t$ with $e_j = +$ we have $p'_{j-1} a_j = p'_j$ and $p''_{j-1} a_j = p''_j$,*
- *and for each $1 \leq j \leq t$ with $e_j = -$ we have $p'_j a_j = p'_{j-1}$ and $p''_j a_j = p''_{j-1}$*

with setting $p'_0 := q_0$ and $p''_0 := q_i$ (See Figure 7).

Then whenever Θ is a reversible congruence on N', the states q_j belong to pairwise different Θ-classes. (In particular, N'/Θ has at least k states.)

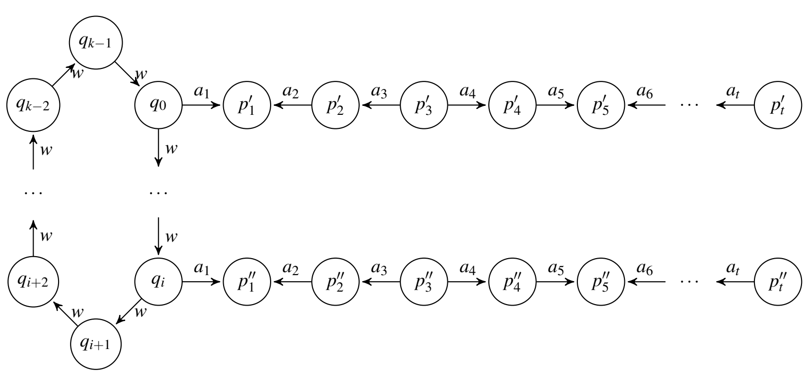


Figure 7: The zig-zag pattern

Before proceeding with the proof, the reader is encouraged to check that the above described pattern appears in the automaton N' of Figure 4 with the choice of $k = 5$, $q_0 = q_{51}$, $q_1 = q_{52}, \ldots, q_4 = q_{55}$, $i = 1$, $w = b$, $t = 3$, $p'_1 = q_2$, $p''_1 = q'_2$, $p'_2 = q_4$, $p''_2 = q'_4$, $p'_3 = q_1$, $p''_3 = q_3$, $a_1 = a$, $a_2 = b$, $a_3 = a$, $e_1 = -$, $e_2 = +$ and $e_3 = -$, with states appearing on the left-hand side of these equations are the states from the pattern of Lemma 2 while states on the right-hand side are states of N'.

Proof. Assume for the sake of contradiction that Θ is a reversible congruence on N' collapsing the states q_ℓ and q_j for some $1 \leq \ell < j \leq k$, that is, $q_\ell \;\Theta\; q_j$.

We claim that $q_{\ell+d} \;\Theta\; q_{j+d}$ for each $d \geq 0$. This holds by assumption for $d = 0$. Using induction on d, assuming $q_{\ell+d} \;\Theta\; q_{j+d}$ we get by applying w that $q_{\ell+d+1} = q_{\ell+d}w \;\Theta\; q_{j+d}w = q_{j+d+1}$ as Θ is a congruence. Hence, writing $j = \ell + \delta$ we get that $q_{\ell+d} \;\Theta\; q_{\ell+\delta+d}$ for each $d \geq 0$, thus in particular for multiples of δ: $q_{\ell+d\cdot\delta} \;\Theta\; q_{\ell+(d+1)\cdot\delta}$.

Hence we have that $q_\ell \;\Theta\; q_{\ell+\delta} \;\Theta\; q_{\ell+2\delta} \;\Theta \ldots$. As k is assumed to be a prime number, there are integers d_1 and d_2 with $\ell + d_1\delta \equiv 0 \bmod k$ and $\ell + d_2\delta \equiv i \bmod k$, thus $q_0 \;\Theta\; q_i$. As p'_0 is defined as q_0 and p''_0 is defined as q_i, we have $p'_0 \;\Theta\; p''_0$.

Now for any integer $d \geq 0$, the relation $p'_d \;\Theta\; p''_d$ implies $p'_{d+1} \;\Theta\; p''_{d+1}$: if $e_{d+1} = +$, then by applying a_{d+1} (since Θ is a congruence), while if $e_{d+1} = -$, then be reversibly applying a_{d+1} (since Θ is a reversible congruence). Hence, it has to be the case $p'_t \;\Theta\; p''_t$ which is nonsense since these two states are assumed to be inequivalent and Θ is a congruence. $\square$

Observe that if some reversible automaton N' recognizing L admits the pattern of Lemma 2 for some prime number k, then there exists a reduced reversible automaton of the form N'/Θ (also recognizing L) which then has at least k states.

In the rest of this part we show that if there exists an irreversible zig-zag state in M, then we can construct such an automaton N' for arbitrarily large primes k, given a reversible automaton N recognizing L. Thus in that case it is clear that there exists an infinite number of reduced reversible automata (up to isomorphism) recognizing L.

First we show that even a weaker condition suffices.

Lemma 3. *Suppose N is a reversible automaton recognizing L such that for each zig-zag state p of M there exist at least two states p' and p'' of N with $p \equiv p' \equiv p''$ and to the ∞-state p_0 of M (of Figure 5), there exist at least k different states $q_0, \ldots, q_{k-1}$ in N, each being equivalent to p_0, with $q_jw = q_{j+1}$ for each $0 \leq j < k$ (again, with $q_k = q_0$).*

Then there exists a reversible automaton N' also recognizing L which admits the zig-zag pattern.

For an example reversible automaton N recognizing L but *not* admitting the zig-zag pattern the reader is referred to Figure 8. We not prove that in these cases the transitions can be "rewired".

Proof. We will construct a sequence $N = N_0, N_1, \ldots, N_t = N'$ of reversible automata, each recognizing L (having the same set of states, and even $(N_i, p) \equiv (N_j, p)$ for each i, j and p, that is, we do not change the languages recognized by any of the states of N) and sequences $p'_0, \ldots, p'_t$ and $p''_0, \ldots, p''_t$ of states such that for each $0 \leq j \leq t$ the following all hold:

- $p'_0 = q_0$, $p''_0 = q_1$,
- if $e_j = +$, then $p'_{j-1}b_j = p'_j$ and $p''_{j-1}b_j = p''_j$ for each automaton N_ℓ with $\ell \leq j$,
- if $e_j = -$, then $p'_jb_j = p'_{j-1}$ and $p''_jb_j = p''_{j-1}$ for each automaton N_ℓ with $\ell \leq j$,

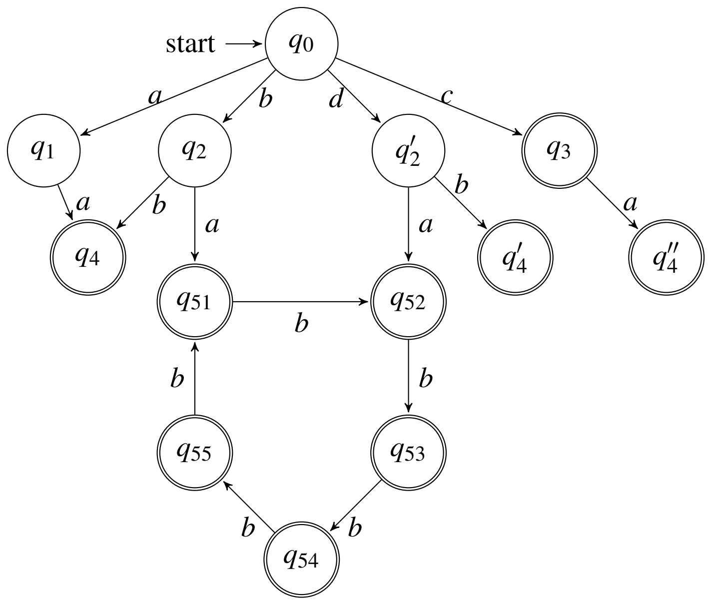


Figure 8: The automaton N does not admit the zig-zag pattern

- if $j < t$ and $\ell \le j$, then p_j is equivalent to both p'_j and p''_j in N_ℓ,
- $p'_t \equiv p_{t,1}$ and $p''_t \equiv p_{t,2}$ in N_t.

If we manage to achieve this, then the automaton $N' := N_t$ indeed admits the zig-zag pattern and still recognizes L.

We construct the above sequence N_j by induction on j. For $j = 0$, choosing $N_0 = N$ and $p'_0 := q_0$, $p''_0 := q_1$ suffices. Having constructed N_j, we construct N_{j+1} based on whether the direction e_{j+1} is $+$ or $-$, the latter one having several subcases.

1. If $e_{j+1} = +$, then let us set $N_{j+1} := N_j$, $p'_{j+1} := p'_j b_{j+1}$ (in N_j) and $p''_{j+1} := p''_j b_{j+1}$ (also in N_j). This choice satisfies the conditions.
2. If $e_{j+1} = -$, then we have three subcases, based on whether there exist states in N_j equivalent to p_{j+1} from which b_{j+1} leads to either p'_j or p''_j.
   (a) If there exist such states r_1 and r_2 with $r_1 b_{j+1} = p'_j$ and $r_2 b_{j+1} = p''_j$, then again, setting $N_{j+1} := N_j$, $p'_{j+1} := r_1$ and $p''_{j+1} := r_2$ suffices.
   (b) Assume there is no such r_1 nor r_2. Then, as p_{j+1} is a zig-zag state, there exist, by the assumption on N, two different states r_1 and r_2, each being equivalent to p_{j+1}. Moreover, as $p_{j+1} b_{j+1} = p_j$ holds, we have that $r_1 b_{j+1}$ and $r_2 b_{j+1}$ (in N_j) are equivalent to p_j. So let us define N_{j+1} as follows: $r_1 b_{j+1} := p'_j$, $r_2 b_{j+1} := p''_j$ and for all the other pairs (r,b) let us leave the transitions of N_j unchanged. Then, setting $p'_{j+1} := r_1$ and $p''_{j+2} := r_2$ suffices.
   (c) Finally, assume that exactly one of these predecessor states exists. By symmetry, we can assume that it is r_1, that is, $r_1 b_{j+1} = p'_j$ in N_j but there is no state r equivalent to p_{j+1} with $r b_{j+1} = p''_j$. Since p_{j+1} is a zig-zag state, there exists some $r_2 \ne r_1$ in N_j, still equivalent to p_{j+1}. In this case we set the transitions in N_{j+1} as $r_2 b_{j+1} p''_j$, leaving the other transitions unchanged suffices with $p'_{j+1} := r_1$ and $p''_{j+1} := r_2$.

$\square$

Hence, given N, it suffices to construct a reversible automaton satisfying the conditions of Lemma 3. First we construct a reversible automaton N' in which for each $\oplus$-state p of M there exist at least two different equivalent states. Let $U \subseteq \Sigma^*$ be the set of words u such that $u = \varepsilon$ or q_0u is *not* a ∞-state. Note that U is a nonempty finite set. Now let us define the state set of N' as the finite set $Q' = Q \times U$, equipped with the following transition function

$$(q,u)\cdot a \;=\; \begin{cases} (qa, ua) & \text{if } qa \text{ is not an } \infty\text{-state,} \\ (qa, u) & \text{otherwise.} \end{cases}$$

For an illustration of the construction starting from N of Figure 4 (a), consult Figure 9.

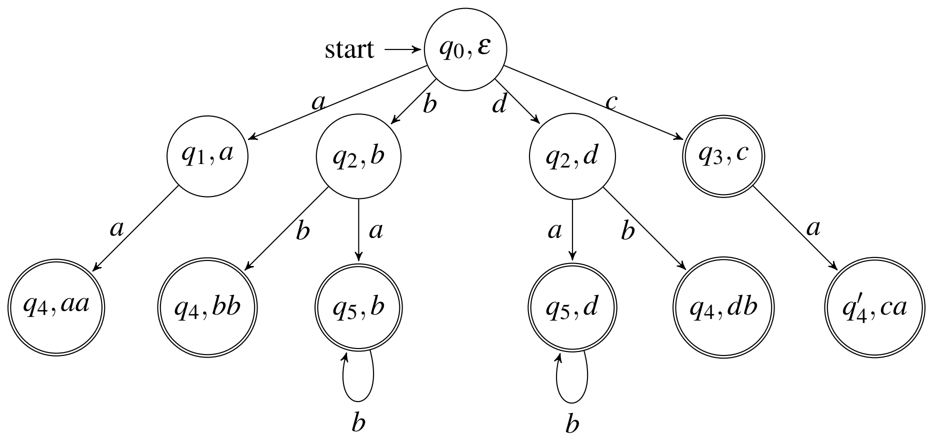


Figure 9: All the $\oplus$-states (namely, q_2 and q_4) have several copies. (Only the trim part of the automaton is shown here.)

It is clear that $(q_0,\varepsilon)u = (q_0u, v)$ for some prefix v of u, moreover, if q_0u is a $\oplus$-state, then $v = u$. Also, states of the form (q,u) in N' are equivalent to q (if we set $F \times U$ as the accepting set). Hence, whenever q is a $\oplus$-state which can be reached by the words $u_1,\ldots,u_n$, $n \geq 2$, then $(q,u_1),\ldots,(q,u_n)$ are pairwise different states in N', reachable from (q_0,ε) and thus to each $\oplus$-state p of M there exist at least two equivalent states in N'. Now starting from N' we will construct an automaton N'' satisfying the conditions of the Lemma.

Let q be a state of N', equivalent to p_0. Since $p_0w = p_0$ in M for the nonempty word w, the sequence q, qw, qw^2,... contains some repetition. Let i be the least integer with $qw^i = qw^j$ for some $j > i$. Then if $i > 0$, then we have $qw^{i-1}\cdot w = qw^i = qw^j = qw^{j-1}w$, thus $qw^{i-1} = qw^{j-1}$ since N' is reversible. Hence $q = qw^j$ for some integer j. In particular, q belongs to a nontrivial SCC of N'. Let av be a shortest nonempty word with $qav = q$ (such a word exists since q is in a nontrivial SCC). For the fixed integer $k \geq 1$, let us define N'' as the automaton over the state set $Q' \times \{0,\ldots,k-1\}$, with transition function

$$(q',i)b \;=\; \begin{cases} (q'b, (i+1) \bmod k) & \text{if } q' = q \text{ and } b = a \\ (q'b, i) & \text{otherwise.} \end{cases}$$

That is, we increase the index i (modulo k) if we get the input a in the state q, and in all the other cases the index remains untouched. For an example with $k = 5$, see Figure 10.

As av is a shortest word with $qav = q$, it is clear that q does not occur on the v-path from qa to $qav = q$. Hence, in N'' we have $(q,i)av = (q,(i+1) \bmod k)$. Moreover, if u is a shortest word in N'

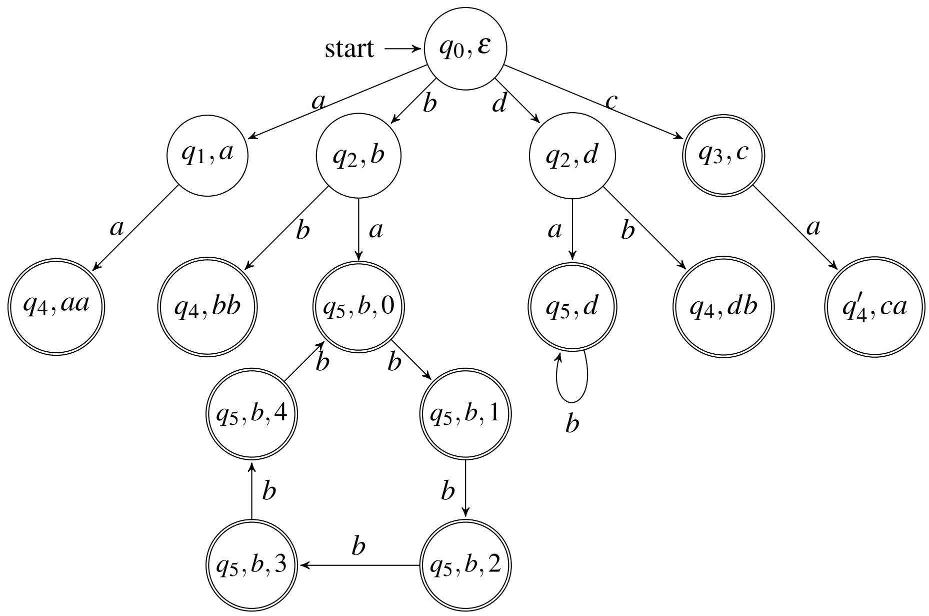


Figure 10: State (q_5,b) is blown up by a factor of $k=5$. (The states inequivalent to q_5 have the index equal to 0 which is not shown here.)

leading into q, then it leads into $(q,0)$ in N''. Thus, in N'' we have the reachable states $q_0=(q,0)$, $q_1=(q,1)$, ..., $q_{k-1}=(q,k-1)$ with $q_iav=q_{(i+1)\bmod k}$ for each i, and still, for each $\oplus$-state p there are at least two different states in N'' equivalent to p. Hence, applying Lemma 3 we get that there exists some reversible automaton N''' (note that the automaton N'' we constructed is also reversible) admitting the zig-zag pattern.

Figure 11 shows the result of this last step: first, since $(q_5,b,0)$ has an incoming a-edge from (q_2,b) but $(q_5,b,1)$ has no such edge, we search for another state equivalent to q_2, that's (q_2,d). Then we set $(q_2,d)a$ to $(q_5,b,1)$. Then, we can follow the b-transitions into (q_4,bb) and (q_4,db) respectively. After that, we should follow a-edges backwards into q_1 and q_3. Hence we rewire the outgoing transitions as $(q_1,a)a=(q_4,bb)$ and $(q_3,c)a=(q_4,db)$ and all is set.

Thus, by Lemma 2 we get the main result of the subsection:

Theorem 3. *If there is an irreversible zig-zag state in M, then for an arbitrarily large k one can effectively construct a reduced reversible automaton equivalent to M, having at least k states.*

Now Theorem 1 is the conjunction of Theorems 2 and 3.

5 Conclusion and acknowledgements

We extended the current knowledge on the reversible regular languages by further analyzing the structure of the minimal automaton of the language in question. In particular, we gave a forbidden pattern characterization of those reversible languages having only a finite number of reduced reversible automata. As the characterization relies on the existence of a forbidden pattern (that of Figure 6), it gives an efficient decision procedure, namely an **NL** (nondeterministic logspace) algorithm: one has to guess a state p_0, then guess some loop from p_0 to itself, then following some back-and-forth walk in the graph of the automaton to two distinct states $p_{t,1}$ and $p_{t,2}$. In the process we also have to check that no 1-state is

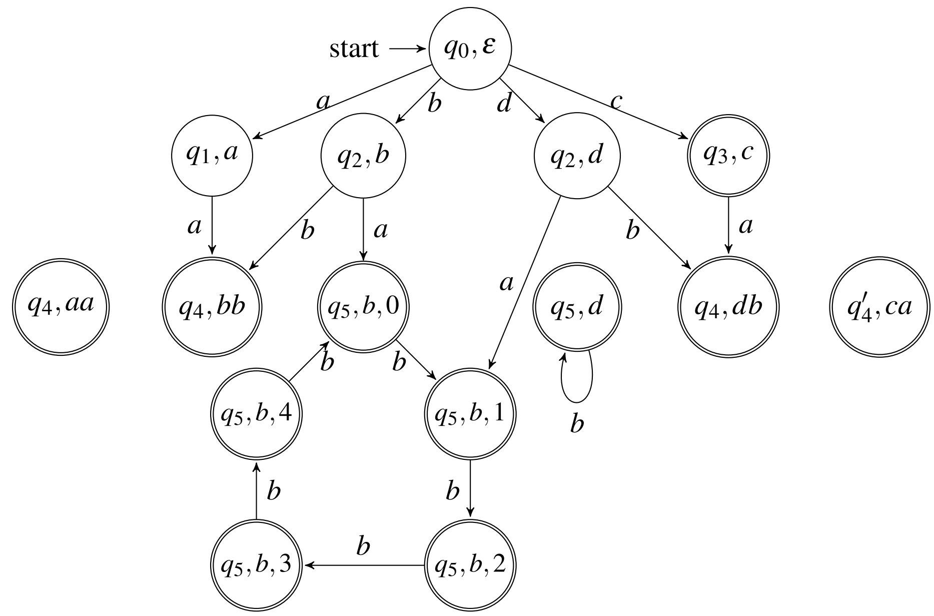


Figure 11: The rewired automaton, admitting the zig-zag pattern. (States (q_4, aa), (q_5, d) and (q'_4, ca) are not part of the resulting trim automaton.)

encountered during this walk (which can clearly also be done in **NL**). It can be an interesting question to study the notion of reduced reversible automata in other reversibility settings, as e.g. in the case of [17]. The authors wish to thank Giovanni Pighizzini, Giovanna Lavado and Luca Prigioniero for their useful comments on a much earlier version of this paper.

References

[1] Dana Angluin (1982): *Inference of reversible languages*. Journal of the ACM 29(3), pp. 741–765, doi:10.1145/322326.322334.

[2] Charles H. Bennett (1973): *Logical reversibility of computation*. IBM Journal of Research and Development 17(6), pp. 525–532, doi:10.1147/rd.176.0525.

[3] Charles H. Bennett (2003): *Notes on Landauer's principle, Reversible Computation and Maxwell's Demon*. Studies in History and Philosophy of Modern Physics 34(3), pp. 501–510, doi:10.1016/S1355-2198(03)00039-X.

[4] Jeffrey Bub (2001): *Maxwell's Demon and the Thermodynamics of Computation*. Studies in History and Philosophy of Modern Physics 32(4), pp. 569–579, doi:10.1016/S1355-2198(01)00023-5.

[5] John Earman & John D. Norton (1999): *The wrath of Maxwell's demon. Part II. From Szilard to Landauer and beyond*. Studies in History and Philosophy of Modern Physics 30(1), pp. 1–40, doi:10.1016/S1355-2198(98)00026-4.

[6] Markus Holzer, Sebastian Jakobi & Martin Kutrib (2015): *Minimal reversible deterministic finite automata*. In: DLT, Lecture Notes in Computer Science 9168, Springer, pp. 276–287, doi:10.1007/978-3-319-21500-6_22.

[7] John E. Hopcroft & Jeffrey D. Ullman (1979): *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley.

[8] Attila Kondacs & John Watrous (1997): *On the power of quantum finite state automata*. In: *FOCS*, IEEE Computer Society, pp. 66–75.

[9] Martin Kutrib (2014): *Aspects of reversibility for classical automata*. In: *Computing with new resources*, *Lecture Notes in Computer Science* 8808, Springer, pp. 83–98, doi:10.1007/978-3-319-13350-8_7.

[10] Martin Kutrib (2015): *Reversible and irreversible computations of deterministic finite-state devices*. In: *MFCS*, *Lecture Notes in Computer Science* 9234, Springer, pp. 38–52, doi:10.1007/978-3-662-48057-1_3.

[11] James Ladyman, Stuart Presnell, Anthony J. Short & Berry Groisman (2007): *The connection between logical and thermodynamic irreversibility*. *Studies in History and Philosophy of Modern Physics* 38(1), pp. 58 – 79, doi:10.1016/j.shpsb.2006.03.007.

[12] Rolf Landauer (1961): *Irreversibility and heat generation in the computing process*. *IBM Journal of Research and Development* 5(3), pp. 183–191, doi:10.1147/rd.53.0183.

[13] Klaus-Jörn Lange, Pierre McKenzie & Alain Tapp (2000): *Reversible space equals deterministic space*. *Journal of Comput. Sys. Sci* 60(2), pp. 354–367 doi:10.1006/jcss.1999.1672.

[14] Giovanna Lavado, Giovanni Pighizzini & Luca Prigioniero (2017): *Weakly and Strongly Irreversible Regular Languages*. In: *AFL*.

[15] Giovanna Lavado & Luca Prigioniero (2017): *Concise representations of reversible automata*. In: *DCFS*, *Lecture Notes in Computer Science* 10316, Springer, doi:10.1007/3-540-10003-2_104.

[16] Giovanna J. Lavado, Giovanni Pighizzini & Luca Prigioniero (2016): *Minimal and Reduced Reversible Automata*. In Cezar Câmpeanu, Florin Manea & Jeffrey Shallit, editors: *Descriptional Complexity of Formal Systems: 18th IFIP WG 1.2 International Conference, DCFS 2016, Bucharest, Romania, July 5-8, 2016. Proceedings*, Springer International Publishing, Cham, pp. 168–179, doi:10.1007/978-3-319-41114-9_13.

[17] Sylvain Lombardy (2002): *On the construction of reversible automata for reversible languages*. In: *ICALP*, *Lecture Notes in Computer Science* 2380, Springer, pp. 170–182, doi:10.1007/3-540-45465-9_16.

[18] Michael Nielsen & Isaac Chung (2000): *Quantum computation and quantum information*. Cambridge University Press.

[19] John D. Norton (2005): *Eaters of the lotus: Landauer's principle and the return of Maxwell's demon*. *Studies in History and Philosophy of Modern Physics* 36, pp. 375–411, doi:10.1016/j.shpsb.2004.12.002.

[20] John D. Norton (2011): *Waiting for Landauer*. *Studies in History and Philosophy of Modern Physics* 42, pp. 184–198, doi:10.1016/j.shpsb.2011.05.002.

[21] Jean-Éric Pin (1992): *On reversible automata*. In: *LATIN*, *Lecture Notes in Computer Science* 583, Springer, pp. 401–416, doi:10.1007/BFb0023844.

[22] Joan Vaccaro & Stephen Barnett (2011): *Information Erasure Without an Energy Cost*. *Proceedings of the Royal Society A* 467(2130), pp. 1770–1778, doi:10.1103/PhysRevLett.102.250602.