

A coordination model for ultra-large scale systems of systems

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The ultra large multi-agent systems are becoming increasingly popular due to quick decay of the individual production costs and the potential of speeding up the solving of complex problems. Examples include nano-robots, or systems of nano-satellites for dangerous meteorite detection, or cultures of stem cells for organ regeneration or nerve repair. The topics associated with these systems are usually dealt within the theories of intelligent swarms or biologically inspired computation systems. Stochastic models play an important role and they are based on various formulations of the mechanical statistics. In these cases, the main assumption is that the swarm elements have a simple behaviour and that some average properties can be deduced for the entire swarm. In contrast, complex systems in areas like aeronautics are formed by elements with sophisticated behaviour, which are even autonomous. In situations like this, a new approach to swarm coordination is necessary. We present a stochastic model where the swarm elements are communicating autonomous systems, the coordination is separated from the component autonomous activity and the entire swarm can be abstracted away as a piecewise deterministic Markov process, which constitutes one of the most popular model in stochastic control.

Keywords: ultra large multi-agent systems, system of systems, autonomous systems, stochastic hybrid systems.

1 Introduction

The ultra large scale systems (ULSS) represent a cross-disciplinary concept that refers to software intensive systems with unprecedented amount of resources and characteristics. The term was defined by Northrop and others in [9] to describe the challenges facing the US Department of Defence.

The systems of systems (SoS) denote complex systems where each part is a system in itself. This area is still evolving, but it is widely accepted that parts of an SoS are systems of systems themselves with some degree of autonomy. Their integration forms a system with more functionality and performance that is more than simply adding the constituent behaviours. Being complex systems, the component systems interact and emergent behaviours appear.

In this paper, we consider aspects of systems of systems at ultra large scale. Specifically, we focus on mathematical modelling and coordination of such systems. We define a simple and abstract model for a component system that we call agent. Each agent can perform a specific activity, which is abstracted away in our model. Then we consider the case of a large number of communicating agents. The main contribution of the paper is to describe the evolution of the SoS as a special Markov process.

The main characteristics of systems studied in this paper are:

- a large, but finite, number of agents;
- elements that can communicate and collaborate;

- agents that are capable of complex behaviour and can be autonomous;
- capability to switch between different operating regimes when they are thought of as systems of systems.

Our approach addresses these issues by employing the following key aspects:

- using hybrid discrete continuous models instead of using the mathematical theory of oscillators;
- the agent autonomy is modelled via hybrid automata;
- the swarm coordination is treated independently from the autonomous behaviour of each agent;
- the analytic tools of statistical mechanics can be used for studying system properties without constraining the models.;
- the system of systems modelling: the individual agent dynamics and the overall aggregate system are characterized at different scales;
- availability of stochastic control techniques by abstracting away the system of systems as a piecewise deterministic Markov process [6], which is a well-established model in control engineering.

2 Problem Formulation

Ultra large scale systems of systems (ULSoS) are composed of a very large number of agents that can interact and coordinate to each other. An agent is understood here as a system that uses a fixed set of rules based on interaction with other agents and information regarding the environment in order to change its internal state and achieve its design objective.

Understanding the causal relation between individual agent characteristics and the collective behaviour represents the major research challenge when dealing with ultra large scale systems of systems. Mathematical modelling and analysis can be used for studying their aggregate dynamics, ergodic behaviour, metastable states, causal relations between individual agents, collective behaviour, and so on.

There exist two fundamentally different approaches for modelling systems of interacting agents. If the number of agents is large, then a continuum population level approach is needed, which will provide some partial differential equations (PDE) for spatially distributed agent densities. The models used are called *Eulerian models*, and they regard the *macroscopic level* (collective behavior) of SoS. The second approach regards the *microscopic level*, and is based on modelling of the given SoS as a system of interactive particles (individual agents). Each particle has its own dynamics and it is subject to specific forces of interaction coming from the other agents. The models used are called *Lagrangian models*, and designing such models might have tremendous consequences for getting desired collective behaviours.

Major challenges in developing analytical frameworks include nonlinearities in the interactions, high dimensionality of the state space, possible randomness due to the environment influences.

In this paper we construct a mathematical framework for the analysis of ULSoS. At the microscopic level, we propose to use stochastic hybrid models to describe the agent dynamics. The interaction of an agent with other agents will be defined via some inputs that will modify the continuous dynamics of the underlying agent. At the macroscopic level, we study a ULSoS as a system of interacting stochastic hybrid systems. For the analysis purposes, such a system of systems needs an appropriate abstraction that is easy to handle and provides also useful insights in the dynamics structure of the given ULSoS. This mathematical framework constitutes an initial basis for developing formal methods for ULSoS [10, 13, 1].

3 Stochastic Hybrid Models

In this section, we present two modelling paradigms for stochastic hybrid systems. The first one is of non-diffusion type and is represented by a class of Markov Processes called Piecewise Deterministic Markov Processes [6]. The second one is of diffusion type, and it is represented by the most general class of stochastic hybrid processes [5]. In fact, the second one is obtained by replacing the continuous deterministic dynamical systems that appear in the description of the first one by diffusion processes. Intuitively, the specifications of the two models are quite similar, but the mathematical apparatus for studying the second class is heavily based on Ito stochastic differential equations.

For the presentation of these models as hybrid automata and comparison with other existing models, the reader is referred to [11].

3.1 Non-Diffusion Models

The most general non-diffusion models for stochastic hybrid systems are represented by Piecewise Deterministic Markov Processes (PDMP) [6]. PDMPs are examples of stochastic hybrid processes with deterministic continuous dynamics in the operation modes. A PDMP is a Markov process (x_t) with two components (q_t, y_t) , where q_t takes values in a discrete set Q and given $q_t = q \in Q$, y_t takes values in an open set $X_q \subset \mathbb{R}^{d(q)}$ for some function $\mathbf{d} : Q \rightarrow \mathbb{N}$. The state space of (x_t) is equal to

$$\mathbf{X} = \{(q, y) | q \in Q, y \in X_q\}.$$

The Borel σ -algebra of \mathbf{X} , denoted by $\mathcal{B}(\mathbf{X})$, is the σ -algebra generated by the open sets. By convention, when referring to sets or functions, “measurable” means “Borel measurable”. Let $\mathcal{P}(\mathbf{X})$ be the space of probability measure on the measurable space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ equipped with the topology of weak convergence. If \mathbf{X} and \mathbf{U} are nonempty topological spaces, a stochastic kernel on \mathbf{X} given \mathbf{U} is a function $R(\cdot, \cdot), R : \mathbf{U} \times \mathcal{B}(\mathbf{X}) \rightarrow [0, 1]$, or $R : \mathbf{U} \rightarrow \mathcal{P}(\mathbf{X})$, such that $R(u, \cdot)$ is a probability measure on \mathbf{X} for each fixed $u \in \mathbf{U}$, and $R(\cdot, B)$ is a measurable function on \mathbf{U} for each fixed $B \in \mathcal{B}(\mathbf{X})$. If \mathbf{X} and \mathbf{U} coincide, R is called stochastic kernel on \mathbf{X} .

In the remainder of this section, we briefly present the realization of a PDMP. Assume that for each point $z = (q, y) \in \mathbf{X}$, there exists a unique, deterministic flow $\phi_q(y, t) \subset X_q$, determined by a differential operator \mathcal{X}_q on $\mathbb{R}^{d(q)}$.

If for some $t_0 \in \mathbb{R}_+$, $z_0 = (q_0, y_0) \in \mathbf{X}$, then y_t , where $t \geq t_0$ follows $\phi_{q_0}(y_0, t)$ until either $t = T_1$ some random time with hazard rate λ or until $y_t \in \partial X_{q_0}$ (the boundary of X_{q_0}). In both cases, the process x_t jumps, according to a probabilistic distribution described by a stochastic kernel R to another location of the state space, $(q_1, y_1) \in \mathbf{X}$. Again, y_t follows a deterministic flow $\phi_{q_1}(y_1, t)$ until a random time T_2 (independent of T_1), or until $y_t \in \partial X_{q_1}$, etc. The jump times T_i are assumed to satisfy the following condition: $\mathbb{E}(\sum_i \mathbf{1}_{T_i \leq t}) < \infty$.

A PDMP is fully described by means of three local characteristics: (i) a global flow $\phi(y_0, t)$ which is the solution of the following ordinary differential equation:

$$\dot{\phi}_q(y_k, t) = b_q(\phi(y_k, t)), t \geq T_k; \phi_q(y_k, T_k) = y_k.$$

(ii) a jump rate $\lambda : \mathbf{X} \rightarrow \mathbb{R}_+$; (iii) a stochastic kernel $R : \bar{\mathbf{X}} \times \mathcal{B}(\mathbf{X}) \rightarrow [0, 1]$.

Let $y_t(x)$ be the sample path of the PDMP with the start point x and $\{(Y_n, T_n) | n = 1, 2, \dots\}$ be the sequence of jump times and corresponding post-jump locations. Between two jumps the evolution represents a deterministic dynamical system given by the flow ϕ and starting with Y_n at time T_n , i.e.

$$y_t(x) = \phi(Y_n, T_n, t), t \in [T_n, T_{n+1}).$$

The post-jump locations have the following probability distributions, for any measurable set $E \in \mathcal{B}(\mathbf{X})$

$$\mathbb{P}_x(Y_{n+1} \in E | T_1, Y_1, \dots, T_n, Y_n, T_{n+1}) = R(\phi(Y_n, T_n, T_{n+1}), E)$$

and the sojourn time in a location (or the time interval between two jumps $S_n = T_{n+1} - T_n$) is given by the following distribution

$$\mathbb{P}_x(S_n \geq t | T_1, Y_1, \dots, T_n, Y_n) = \exp\left\{-\int_{T_n}^{t+T_n} \lambda(y(x, s)) ds\right\}.$$

The resulting process is a Borel right process [6], i.e., a particular strong Markov process with some additional properties regarding the state space and the continuity of the trajectories.

3.2 Diffusion Models

For the purposes of this paper, in the following we give a simplified version of the general model of stochastic hybrid systems presented in [5]. A stochastic hybrid system (SHS) is a Markov process (x_t) with two components (q_t, z_t) , where q_t takes values in a discrete set Q and given $q_t = q \in Q$, y_t takes values in an open set $X_q \subset \mathbb{R}^{d(q)}$ for some function $\mathbf{d} : Q \rightarrow \mathbb{N}$. The state space of (x_t) is equal to $\mathbf{X} = \{(q, y) | q \in Q, y \in X_q\}$. Usually, the state space \mathbf{X} is embedded in an Euclidean space \mathbb{R}^n . The closure $\bar{\mathbf{X}}$ can be partitioned into a boundary \mathbf{X}_δ and interior \mathbf{X}_o , that will play an important role in defining the hybrid behaviour. The boundary \mathbf{X}_δ will play the role of guards from the classical hybrid automata modelling.

Under standard assumptions an SHS can be uniquely characterized by: (i) a vector field: $b : \mathbf{X} \rightarrow \mathbb{R}^d$, (ii) a matrix: $\sigma : \mathbf{X} \rightarrow \mathbb{R}^{d \times m}$ that is a \mathbb{R}^d -valued matrix, $m \in \mathbb{N}$, (iii) an intensity function or jump rate: $\lambda : \mathbf{X} \rightarrow \mathbb{R}_+$, and (iv) stochastic kernels: $R_o : \mathbf{X}_o \rightarrow \mathcal{P}(\mathbf{X})$, and $R_\delta : \mathbf{X}_\delta \rightarrow \mathcal{P}(\mathbf{X})$.

In each mode X^q , the continuous evolution is driven by the following stochastic differential equation (SDE)

$$dz_t^q = b(q, z_t^q)dt + \sigma(q, z_t^q)dW_t, \quad (1)$$

where $(W_t, t \geq 0)$ is the m -dimensional standard Wiener process in a complete probability space. The discrete component remains constant, i.e., $q_t = q$.

In the interior of the state space \mathbf{X}_o , the process may have discrete transitions with the rate $\lambda(x)$ when the process is at state x , independently of the process history. Then the process is transferred immediately to a new state randomly according to the stochastic kernel $R_o(x|dx)$. This type of discrete transition is called *spontaneous transition*. If the process reaches the boundary at $x \in \mathbf{X}_\delta$, the process has a discrete transition to a new random state given by $R_\delta(x|dx)$. This type of discrete transition is called *forced transition*.

Always, we assume that $R_o(x, \mathbf{X}_o) = 1$ and $R_\delta(x, \mathbf{X}_o) = 1$.

Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $0 \leq T_1 < T_2 < \dots < T_i < T_{i+1} < \dots$ be the sequence of jump times. The resulting process is a Borel right process as in the case of PDMPs.

4 Microscopic Level

Let us consider an ULSoS of agents, whose behaviour exhibits discrete and continuous dynamics with uncertainty features. Suppose that the ULSoS has a large number of agents, each one having the dynamics described by a diffusive-type model of stochastic hybrid system (that will be described below). In this paper, we consider the case when the agent interactions will change the hybrid structure of its behaviour by keeping the original discrete transitions, but adding new discrete transitions as result of the alteration of the continuous dynamics. More specifically, the continuous dynamics for an agent mode can be modified using inputs coming from other agents, i.e., it might encounter *new discrete transitions* dictated by these interactions. Then one operational mode is split in some new modes resulted from the interaction between the hybrid agent and the entire collective. To add more flexibility to the models of stochastic hybrid systems used for the agent modelling, we consider that the mode boundaries (guards) are not fixed in time. To achieve this, we need to allow guards that exhibit dynamics governed by some ordinary differential equations (ODE).

4.1 Hybrid Agent Model

The mathematical model for a hybrid agent is a stochastic hybrid system with a peculiar structure. The system has two types of discrete transitions:

- event triggered transitions, which are generated by the detection of certain events. In the context of the massively parallel collective, these events are generated by the inter-agent communication, when an input message is received.
- forced transitions, which are triggered by guards that can evolve in time.

The system has input output activities, which follow a certain communication policy. An output is sent only when a forced discrete transition takes place. The inputs are collected only during continuous evolutions.

The continuous evolution of a hybrid agent has also a parallel structure by executing simultaneously two distinct modes. Let us call these two modes as the *coordination*, respectively the *activity* mode. These modes start and stop synchronously. The coordination mode is, in fact, a hybrid system by itself. Every input generates a discrete jump, and modifies the dynamics in the coordination mode.

Formally, the activity of each agent is described as a stochastic hybrid process $\mathbf{M}^i = (q_t^i, z_t^i, u_t^i)$ (viewed as a sort of revival process) defined on the hybrid state space $\mathbf{X}^i \times \mathbf{U}^i$, with some ‘active’ (time depend) guards defined on \mathbf{X}^i . Note the definition of \mathbf{X}^i should be slightly different with respect to the classical case when the boundaries are fixed. To avoid undesired complications, we suppose that for all i , the hybrid state spaces \mathbf{X}^i can be embedded in the Euclidean space \mathbb{R}^d . The pair (q_t^i, z_t^i) will be called the coordination component, and the pair (q_t^i, u_t^i) will be called activity component. The guards are defined as ‘active boundaries’ or thresholds $(\partial_t^i) := (q_t^i, \beta_t^i)$ that replace the fixed boundaries from the standard definition of SHS. More precisely, in the absence of interactions, between the jump times, the process follows the dynamics law given by some stochastic differential equations (1). The jumping times are defined as hitting times of the active boundaries. Therefore, the evolution of such a hybrid system will be described by the tuple $(q_t^i, z_t^i, \beta_t^i), t \geq 0$, which is a right continuous stochastic process on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For each agent i , it is assumed that $z_t^i < \beta_t^i$ (the order is defined componentwisely in the Euclidean space), for all $t \geq 0$, except for the jumping moments of time $0 < T_1^i < T_2^i < \dots$, when $z_{(T_k^i-)}^i = \beta_{(T_k^i-)}^i$.

The active boundary is thought of as a moving barrier $\{\beta_t^i | t \geq 0\}$, with $\beta_0^i = \gamma^i$. The jumping times are defined as the first hitting times of the moving barrier by the continuous process (z_t^i) . The dynamics of the barrier will be given by a simple first order differential equation:

$$\frac{d\beta^i}{dt} = F^i(\beta^i), t > 0, \beta_0^i = \gamma^i. \quad (2)$$

The active boundary dynamics $\varphi^i(t)$, which is a curve defined respectively for each mode of the underlying agent i , is the solution of (2). Based on the hybrid nature of the underlying system, we can think that the moving barrier is defined piecewisely for each mode. Then, we can refine (2) accordingly. In practice, we need to consider particular classes of ODE to define the barrier dynamics, such that moments or probability distributions of the jumping times can be analytically or numerically computed. Considering the computation difficulty of the first time passage problem when the active barrier has a quite general form (see [12]), we specialize (2) such that

$$\frac{d\beta^i}{dt} = (-k^i) \cdot \beta^i, t > 0; \beta_0^i = \gamma^i. \quad (3)$$

Then, the active boundaries have an exponential form $\beta_t^i = \gamma^i e^{-k^i t}$ such that the computation of the expectations of the jumping times admits numerical solutions.

In the initial hybrid model of the agent i , we can also impose a reset condition for the boundary variable $\partial = (q, \beta)$, as a stochastic kernel

$$R_{\partial}^i : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \rightarrow [0, 1].$$

The role of R_{∂}^i is to provide the probability law for the initial condition γ^i of the ODE that governs the guard dynamics.

Intuitively, an agent can have an independent evolution, or one which is coordinated with the collective. When it evolves independently, the agent executes an activity for a relatively long time until a forced transition takes place and the agent switches to a different activity. For example, the activities can be navigation under a certain direction, or rest. More sophisticated scenarios could include the detection of a malign tumour, or drug delivery. Different views can be used in defining activities. These can be just simply labels from a given finite set, or they can be modelled into more details by differential equations. One can add more details by considering noise or other perturbations, and the mathematical model changes into a stochastic differential equation. The collective behaviour requires a formal coordination mechanism. This mechanism consists of perturbations of the dynamics leading to a forced transition, generated by other agents via communication. In order to keep the model simple, we make the assumption that the inter-agent communication is not affecting directly an agent activity. Instead, only the dynamics in the coordination mode is affected. Since the guards of the forced transitions are related to the dynamics in the coordination mode, the communications can speed up the execution of a forced transition (and in this way make an activity change). A single communication may not trigger a forced transition. Some times a repeated communication from a single agent, or communications from several agents are necessary. Equally, a communication can speed or slow the process of executing a forced transition. These aspects are relevant for the problem of stability, which is not treated in this paper.

The communication takes place along a bidirectional channel. Every hybrid agent communicates only with a finite number of other agents called its neighborhood. Each communication consists of a single bit. Practically, when an agent executes a forced transition, all its neighbors are announced about that.

Let formally model this sort of interaction between the individuals of a ULSoS.

For each agent i , let define $\Upsilon_t^i := t - T_k^i$ if $t \in [T_k^i, T_{k+1}^i)$, for $t \geq 0$. Υ_t^i denotes the time elapsed since the last forced jump of the i th agent until the moment t . Clearly, the discrete state is constant between jumps, i.e., $q_t^i = q_{T_k^i}^i$ if $t \in [T_k^i, T_{k+1}^i)$.

Remark 1 When $t \in [T_k^i, T_{k+1}^i)$, the time $\Upsilon_t^i(\omega)$ can be thought of as a local clock for that part of the trajectory ω that lies in the mode $q_{T_k^i}^i$. Then there is a one-to-one correspondence between the trajectories of the hybrid agent i and the trajectories of (Υ_t^i) .

The dynamics of the agent i , for the coordination mode $q_{T_k^i}^i$, i.e., $t \in [T_k^i, T_{k+1}^i)$, is hybrid discrete continuous. Within the interval between two consecutive communication events, the dynamics is continuous. When a communication event takes place, a discrete transition is produced, and then the continuous dynamics changes. Let us formulate the analytics of this process. To each unidirectional communication along the channel between the agents i and j , we associate a characteristic vector $w^{ij} \in \mathbb{R}^d$.

For each agent i , let us consider the overall changes in the dynamics due to all communications with its neighbors I_t^i given by

$$I_t^i := \sum_{j \in N^i} w^{ij} \exp(-k^i \Upsilon_t^i),$$

where N^i is the neighborhood of the agent i , i.e.

$$N^i := \{j : |w^{ij}| \geq w; \Upsilon_t^j \leq \Upsilon_t^i\},$$

where $w > 0$ is a lower threshold for the strength of interaction.

After all communications took place, the dynamics in the coordination mode is given by the following equation

$$\tilde{z}_t^i := z_{\Upsilon_t^i}^i + I_t^i, t \in [T_k^i, T_{k+1}^i).$$

Now, we explain how the dynamics changes after each communication via a recursive process where we define (\tilde{z}_t^i) recurrently, as follows. Suppose that

$$T_k^i \leq T_{k_1}^{j_1} < T_{k_2}^{j_2} < \dots < T_{k_p}^{j_p} \leq t < T_{k+1}^i.$$

Then

$$\begin{aligned} (\tilde{z}_t^i)_1 & : = z_{\Upsilon_t^i}^i + w^{ij_1} \exp(-k^i \Upsilon_t^i), \\ (\tilde{z}_t^i)_r & : = (\tilde{z}_t^i)_{r-1} + w^{ij_r} \exp(-k^i \Upsilon_t^i); r = 2, \dots, p. \end{aligned}$$

It is clear that each jump of the external agent j (with respect to the agent i) enables a jump in the continuous dynamics of agent i of length w^{ij} . Roughly speaking, I_t^i is forcing the apparition of some discrete transition of the agent i due to the communication with the agents that have already exhibited such transitions.

Figure 1 illustrates the evolutions of a one-dimensional hybrid agent. The forced transitions are marked by a vertical line. The horizontal line marks the asymptotic limit of the dynamic guard. A forced transition is triggered when the guard and the dynamics in the coordination mode reach a common value. One can easily remark the jumps determined by the communication events.

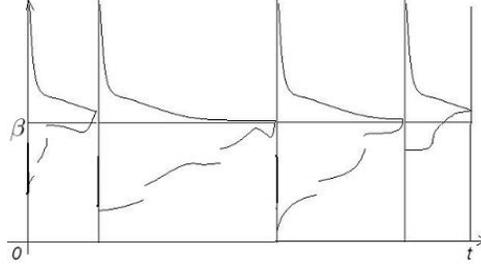


Figure 1: Evolution of a hybrid agent

4.2 Equivalent Description

The first passage time of the active boundary corresponding to the stochastic hybrid process (z_t^i) can be thought also as the first hitting time of a modified boundary corresponding to the initial process (z_t^i) . The new boundary continuous variable can be obtained as:

$$\bar{\beta}_t^i := \beta_t^i - I_t^i. \quad (4)$$

This means that the interactions between agents can be thought also as acting on guards: The law of the active boundary of an agent is changed according with the inputs coming from the other agents when they have a jump. In other words, a change of the operational mode of one hybrid agent in the ULSoS influences the guards (in our case, the active boundaries) of the other agents. This new perspective on the type of interaction that we have already defined here will help us for developing the analytical tools for the macroscopic level of a ULSoS.

To capture the complexity of a one agent dynamics and its interactions with other agents, we need to consider also the time process Υ_t^i , i.e.

$$(a_t^i) := (q_t^i, \tilde{z}_t^i, \beta_t^i, \Upsilon_t^i), \quad t \geq 0. \quad (5)$$

Due to the interaction with the other agents, (a_t^i) is not necessarily a Markov process. However, it becomes a Markov process if there are no interactions, or if we fix Υ_t^j for the agents $j \in N^i$. Moreover, the evolution of the process (q_t^i, \tilde{z}_t^i) can be encoded in the evolution of $(q_t^i, \beta_t^i, \Upsilon_t^i)$. Then the agent activity will be driven only by the boundary dynamics and the jumping times encapsulated in Υ_t^i , i.e., $(\partial_t^i, \Upsilon_t^i)$. A similar idea has been used in the use of jump processes for studying Piecewise Deterministic Markov Processes (PDMP) - see [6]. Note that in our case $(\partial_t^i, \Upsilon_t^i)$ is not a jump process, but it is a hybrid process with deterministic continuous dynamics.

Let ζ^i be the first hitting time of (z_t^i) to reach the curve $(\bar{\beta}_t^i)$ defined by (4). Let us collect all

- N clock variables as $\tau := (\tau^1, \dots, \tau^N)$,
- N guard variable as $\beta := (\beta^1, \dots, \beta^N)$ (note that β is an $N \times d$ dimensional vector).

Define $\tau^{(-i)} := (\tau^1, \dots, \tau^{i-1}, \tau^{i+1}, \dots, \tau^N)$. Let I be a measurable set of $[0, \infty)$, and let us define (as a conditional probability) the following measure:

$$\mu_\beta^i(I) := \mathbb{P}[\zeta^i \in I | \tau^{(-i)}, \beta].$$

Suppose that there is a probability density function associated to μ_β^i , i.e., $\psi^i(t|\tau^{(-i)}, \beta)dt := \mu_\beta^i(dt)$ and define Ψ^i as the probability that the stopping time ζ^i is less than τ^i , i.e.,

$$\Psi^i(\tau^i|\tau^{(-i)}, \beta) := \mu_\beta^i([0, \tau^i]) = \int_0^{\tau^i} \psi^i(t|\tau^{(-i)}, \beta)dt$$

Theorem 1 $(\beta_t^i, \Upsilon_t^i)$ is a Piecewise Deterministic Markov Process.

Proof. The standards features that characterize a PDMP are: deterministic dynamics for the continuous evolution, discrete transitions (governed either by a rate function, or by guards), and a reset map defined as stochastic kernel. In our case, it is clear that the continuous dynamics is governed by simple ODEs. The process Υ_t^i is just simply increasing with the unit rate, and has reset to zero whenever z_t^i has a jump. Moreover, our process $(\beta_t^i, \Upsilon_t^i)$ does not have forced jumps (due to the existence of guards). The discrete transitions take place in a Poisson type fashion with respect to a rate function. We identify this rate function as the following measurable bounded function

$$\lambda^i(\tau, \beta) := \frac{\psi^i(\tau^i|\tau^{(-i)}, \beta)}{1 - \Psi^i(\tau^i|\tau^{(-i)}, \beta)}. \quad (6)$$

This has the role of a transition rate: the probability that the agent i has a jump in the interval Δt is equal to $\lambda^i(\tau, \beta)\Delta t + o(\Delta t)$. The probability that in the interval Δt , two or more agents may have discrete transitions is $o(\Delta t)$. The reset kernel is trivial (R_∂^i, R_0^i) , where R_∂^i is the reset kernel for the guard, and R_0^i is the reset to 0 of the clock variable. ■

Remark 2 $(\beta_t^i, \Upsilon_t^i)$ is a PDMP with spontaneous discrete transitions governed by $\lambda^i(\tau, \beta)$, but no forced discrete transitions.

5 Macroscopic Level

The ULSoS collective behaviour is described by the interactive superposition of its hybrid agents $(q_t^i, z_t^i, \beta_t^i, \Upsilon_t^i)$, $i = 1, \dots, N$. We denote this superposition as follows

$$\otimes^\diamond(q_t^i, z_t^i, \beta_t^i, \Upsilon_t^i)$$

The entire ULSoS activity is completely described by the embedded Markov hybrid process (∂_t, Υ_t) defined on the hybrid state space obtained as the superposition of the agent state spaces $\bigcup_{q \in Q^i} \{q\} \times \mathbb{R}^d \times [0, \infty)$. In fact, the boundary variable β is carrying in the structure also information about the discrete state q . Therefore, to simplify the up-coming analysis, we need only the process (β_t, Υ_t) defined on $\mathbb{R}^{d \times N} \times [0, \infty)^N$. Such a process play the role of a macroscopic ‘abstraction’ for the ULSoS behaviour. The executions of the process (β_t, Υ_t) can be described as follows. Each component β_t^i of β_t follows the dynamics described by (3) in the interval between two forced transitions of the agent i , whereas each component Υ_t^i of Υ_t follows a trivial ODE with the rate 1. As an easy consequence of the Remark 1, we get the following result.

Theorem 2 For any initial condition, there is a one-to-one correspondence between the sample paths of $\otimes^\diamond(q_t^i, z_t^i, \beta_t^i, \Upsilon_t^i)$ and (β_t, Υ_t) .

Proof. The proof can be done in the same style used by Davis [6] to prove that there is a one-to-one correspondence between the paths of a jump process and the paths of a PDMP. ■

The process (β_t, Υ_t) will be called *ULSoS abstraction*. We need to clarify this concept, because all the further developments of this work are based on this. Recall that at the microscopic level we have modelled the agents hybrid behaviour and their interactions using a sort of communicating stochastic hybrid systems. Then the collective behaviour of the ULSoS is a complex stochastic hybrid system obtained by sticking together the agent dynamics described by (5). The analysis of this new hybrid system requires indeed complicated mathematics, since we have to consider different facets: stochastic differential equations, discrete transitions governed by dynamic guards, interaction between agents that might change the continuous dynamics, and so on. The observation that from all parameters that describe the ULSoS behaviour, only two of them can be used to construct a sort of skeleton process that characterizes the entire ULSoS dynamics is essential for this analysis. This point is crucial for developing our approach for finding useful characterizations of the ULSoS at the macroscopic level. Moreover, we can go further with the abstraction process and to derive from the given PDMP the embedded Markov jump process.

For a given collective, one can designate some hybrid agents to play the role of input, and similarly some output agents. In this way, the collectives can be composed by connecting one collective's output to the input of the second. At the macroscopic level, the composite collective is described by sequential composition of PDMPs. Various composition operators for PDMPs form the so-called process algebra. This has been developed in [4]. This is the key for the modular development of multi-agent collectives.

For the macroscopic description of the ULSoS, it is necessary to provide some PDE to describe the dynamics of the spatially distributed agent densities. In our case, the abstraction process (β_t, Υ_t) is embedded in the dynamics structure of the entire ULSoS. This abstraction process is bidirectional: some properties of the abstraction process will characterize also the whole ULSoS dynamics, but also the agent dynamics. For the ULSoS modelling framework we have defined here, the PDE that will arise naturally as backward Kolmogorov equation, or forward Kolmogorov (Fokker-Planck) equation associated to the ULSoS abstraction. Such equations will describe the evolution of the guards and local clocks probability distributions. The first step for the derivation of such equations is to obtain the mathematical expression of the *infinitesimal generator* associated to (β_t, Υ_t) .

5.1 Infinitesimal Generator

Let us briefly recall the concept of infinitesimal generator. Suppose that (x_t) is a Markov process with an homogeneous transition probability function $(p)_{t \geq 0}$. For each $t \geq 0$, define conditional expectation operator by

$$\mathbf{P}_t f(x) := \int f(y) p_t(x, dy) = \mathbb{E}_x f(x_t), \forall x \in \mathbf{X}; \quad (7)$$

where \mathbb{E}_x is the expectation with respect to \mathbb{P}_x . Here, f belongs to $\mathcal{B}_b(\mathbf{X})$, which is the lattice of all bounded measurable real functions defined on \mathbf{X} . The Chapman-Kolmogorov equation guarantees that the linear operators \mathbf{P}_t satisfy the semigroup property: $\mathbf{P}_{t+s} = \mathbf{P}_t \mathbf{P}_s$. This suggests that the semigroup of (conditional expectation) operators $\mathcal{P} = (\mathbf{P}_t)_{t \geq 0}$ can be considered as a sort of parameterization for a Markov process.

Associated with the semigroup (\mathbf{P}_t) is its *infinitesimal generator* which, loosely speaking, is the derivative of \mathbf{P}_t at $t = 0$. Let $D(L) \subset \mathcal{B}^b(\mathbf{X})$ be the set of functions f for which the following limit exists

$$\lim_{t \searrow 0} \frac{1}{t} (\mathbf{P}_t f - f) \quad (8)$$

and denote this limit Lf . The limit refers to convergence in the supnorm $\|\cdot\|$ of the Banach space $\mathcal{B}_b(\mathbf{X})$, i.e. for $f \in D(L)$ we have:

$$\lim_{t \searrow 0} \left\| \frac{1}{t} (\mathbf{P}_t f - f) - Lf \right\| = 0.$$

For a PDMP defined as in Section 3.1, the infinitesimal generator has the following expression

$$\mathcal{L}f(x) = b(x) \cdot \nabla f(x) + \lambda(x) \int [f(y) - f(x)] R(x, dy) \quad (9)$$

for any $f \in D(\mathcal{L})$. The domain of the generator $D(\mathcal{L})$ is fully described in [6].

Now, coming back to our process (β_t, Υ_t) , it is clear, from the construction, that this new process is obtained by the interacting PDMP components. Then, the expression of its infinitesimal generator will be based on the well known generator expression for PDMP.

For a better understanding, we derive first the expression of the infinitesimal generator corresponding to a hybrid agent viewed as a PDMP.

Proposition 3 *The infinitesimal generator associated to $(\beta_t^i, \Upsilon_t^i)$ maps a continuous differentiable function $f^i(\beta^i, \tau^i): \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$ as follows:*

$$\begin{aligned} \mathcal{L}^i f^i(\beta^i, \tau^i) &= \lambda^i(\beta^i, \tau^i) \int_{\mathbb{R}^d} (f^i(\theta^i, 0) - f^i(\beta^i, \tau^i)) R_{\theta}^i(\beta^i, d\theta^i) \\ &\quad + (-k^i) \sum_{p=1}^d \beta_p^i \frac{\partial f^i}{\partial \beta_p^i}(\beta^i, \tau^i) + \frac{\partial f^i}{\partial \tau^i}(\beta^i, \tau^i). \end{aligned} \quad (10)$$

Proof. Applying directly the general formula (9), with the reset kernel $R_{\theta}^i \otimes R_0^i$. The effect of R_0^i is the apparition of 0 in $f^i(\theta^i, 0)$ in the integral part of (10). ■

Remark 3 *If one would consider to work with more general dynamics (2), then the differential part of the generator expression (10) has to be changed accordingly with the more general formula (9).*

Let us define the vector field $b: \mathbb{R}^{N \times (d+1)} \rightarrow \mathbb{R}^{N \times (d+1)}$ that describe the continuous evolution of (β_t, Υ_t) as follows:

$$b := (-k^1 \beta_1^1, \dots, -k^1 \beta_d^1, \dots, -k^N \beta_1^N, \dots, -k^N \beta_d^N, 1, 1, \dots, 1).$$

Given a function $f \in \mathcal{C}^1(\mathbb{R}^{N \times (d+1)}, \mathbb{R})$ and a vector field b , we use $\mathcal{L}_b f$ to denote the *Lie derivative* of f along b given by

$$\begin{aligned} \mathcal{L}_b f(\beta, \tau) &= \sum_{p=1}^{N \times (d+1)} \frac{\partial f}{\partial (\beta, \tau)_p}(\beta, \tau) b_p(\beta, \tau) \\ &= \sum_{i=1}^N (-k^i) \sum_{p=1}^d \beta_p^i \frac{\partial f}{\partial \beta_p^i}(\beta, \tau) + \sum_{i=1}^N \frac{\partial f}{\partial \tau^i}(\beta, \tau) \end{aligned}$$

where

$$\begin{aligned} (\beta, \tau) &: = (\beta^1, \beta^2, \dots, \beta^N, \tau); \\ \beta^1 &: = (\beta_1^1, \dots, \beta_d^1); \beta^2 := (\beta_1^2, \dots, \beta_d^2) \\ &\dots \\ \beta^N &: = (\beta_1^N, \dots, \beta_d^N); \tau := (\tau^1, \dots, \tau^N). \end{aligned} \quad (11)$$

Let us define a stochastic kernel R_∂ obtained by the superposition of the corresponding kernels for all agents, i.e.

$$\begin{aligned} R_\partial &: \mathbb{R}^{N \times d} \times \mathcal{B}(\mathbb{R}^{N \times d}) \rightarrow [0, 1] \\ R_\partial &: = R_\partial^1 \otimes R_\partial^2 \otimes \dots \otimes R_\partial^N. \end{aligned}$$

Now we have all the elements to write down the infinitesimal generator of (β_t, Υ_t) .

We use the notation $\beta^i(\theta)$ to express the fact that, in the expression (11), the component β^i has been replaced by θ , and the notation $\tau^i(0)$ to say that, in the same expression, the element τ^i has been replaced by 0.

Theorem 4 *The infinitesimal generator associated to (β_t, Υ_t) can be expressed as follows:*

$$Lf(\beta, \tau) = L_{cont}f(\beta, \tau) + L_{jump}f(\beta, \tau) \quad (12)$$

where

$$L_{cont}f(\beta, \tau) := \mathcal{L}_b f(\beta, \tau),$$

and

$$L_{jump}f(\beta, \tau) := \sum_{i=1}^N \left\{ \lambda^i(\beta, \tau) \cdot \int_{\mathbb{R}^{d \times N}} (f(\beta^i(\theta), \tau^i(0)) - f(\beta, \tau)) R_\partial^i(\beta^i, d\theta) \right\}.$$

Proof. (β_t, Υ_t) is a PDMP obtained by the interaction of the PDMP components. Then the expression of the infinitesimal generator follows the general expression of a PDMP generator, taking also into account the interacting factors. ■

Remark 4 *The infinitesimal generator of the ULSoS abstraction is obtained by summing the generators of the PDMP components. The interaction between the components is captured only by the transition rates $\lambda^i(\beta, \tau)$.*

5.2 PDE Characterizations

Departing from the expression of the infinitesimal generator of the ULSoS abstraction, one can obtain the PDE associated. In the following, we give a short background on the Kolmogorov equations associated to a Markov process, and then explain the peculiarities of such equations for stochastic hybrid systems, and, in particular, for ULSoS.

5.2.1 Kolmogorov Equations for Markov Processes

This subsection recalls some basic facts concerning the backward and forward Kolmogorov equation for Markov processes. The forward equation is also known as the Fokker Planck Kolmogorov (FPK) equation for diffusion processes. The Fokker Planck equation is one of the basic tools when dealing with diffusion processes, because it allows to calculate the probability density function (pdf) ρ_t of the process at time $t \geq 0$ given an initial probability density ρ_0 and eventually the stationary pdfs (when they exist).

The semigroup (\mathbf{P}_t) of a Markov process $\mathbf{M} = (x_t)$ satisfies the following differential equation: for all $f \in D(L)$,

$$\frac{d}{dt} \mathbf{P}_t f = L \mathbf{P}_t f. \quad (13)$$

This equation is called *Kolmogorov's backward equation* [6]. In particular, if we define the function $u(t, x) = \mathbf{P}_t f(x)$ then u is solution of the PDE

$$\begin{cases} \frac{\partial u}{\partial t} = Lu \\ u(0, x) = f(x). \end{cases}$$

Conversely, if this PDE admits a unique solution, then its solution is given by $\mathbf{P}_t f(x)$. Moreover, it is easy to check that the operators \mathbf{P}_t and L commute. Then (13) may be written as

$$\frac{d}{dt} \mathbf{P}_t f = \mathbf{P}_t L f. \quad (14)$$

This equation is known as *Kolmogorov's forward equation*. It is the weak formulation of the equation $\frac{d}{dt} \mu_t^x = L^* \mu_t^x$, where the probability measure μ_t^x on \mathbf{X} denotes the law of (x_t) conditioned on $x_0 = x$ and where L^* is the adjoint operator of L .

In particular, if \mathbf{M} is a diffusion process on \mathbb{R}^n and if $\mu_t^x(dy)$ admits a density $q(x; t, y)$ with respect to the Lebesgue measure, the forward Kolmogorov equation is the weak form (in the sense of distribution theory) of the PDE

$$\frac{\partial}{\partial t} q(x; t, y) = - \sum_{i=1}^n \frac{\partial}{\partial y_i} (b_i(y) q(x; t, y)) + \sum_{i,j=1}^d \frac{\partial^2}{\partial y_i \partial y_j} (w_{ij}(y) q(x; t, y)), \quad (15)$$

where $b_i(x)$ and $w_{ij}(x)$ are respectively the drift coefficient and the diffusion coefficient of the process. This equation is known as the Fokker-Planck equation associated to a diffusion process.

5.2.2 Kolmogorov Equations for ULSoS

The macroscopic description of a ULSoS is described by a Markov jump type process. Here, jump process is understood in a rather large sense, i.e. process with discontinuities in the natural filtration. A complete description of a Markov jump process is given by its transition density function, which is the solution of the forward and backward Kolmogorov equations.

A generalized Fokker Planck equation is well known for the case of switching diffusions (where there are no forced transitions). A unifying formulation of the Fokker-Planck-Kolmogorov equation for general stochastic hybrid systems is developed in [2]. For some particular PDMPs, FPK equation has proved to be an useful tool for studying multi-agent systems [8].

The FPK equation for stochastic hybrid systems is based on the concept of mean jump intensity. Let us define a positive measure J on $\mathbf{X} \times (0, \infty)$ by

$$J(A) = \mathbb{E}_{\mu_0} \left\{ \sum_{k \geq 0} 1_A(x_{T_k}^-, T_k) \right\}.$$

For any $\Gamma \in \mathcal{B}$, the quantity $J(\Gamma \times (0, t])$ is the expected number of jumps starting from Γ during the interval $(0, t]$.

Suppose that there exists a mapping $r : t \mapsto r_t$, from $[0, \infty)$ to the set of all bounded measures on \mathbf{X} such that for all $\Gamma \in \mathcal{B}$, we have: (a) $t \mapsto r_t(\Gamma)$ is measurable; (b) for all $t \geq 0$,

$$J(\Gamma \times (0, t]) = \int_0^t r_l(\Gamma) dl.$$

Then r is called the *mean jump intensity* of the process \mathbf{M} under the initial law μ_0 .

The generalized FPK equation can be written symmetrically as

$$\mu_t' = \mathcal{L}_{cont}^* \mu_t + \int (W_t(dx, \cdot) - W_t(\cdot, dx)) \quad (16)$$

where $W_t(dx, dy) = r_t(dx)R(x, dy)$ (R is the stochastic kernel that provides the probability distributions of the post jump locations), or

$$\mu'_t = \mathcal{L}_{cont}^* \mu_t + r_t(R - I) \quad (17)$$

where I is the identity kernel, i.e. $I(x, dy) = \delta_x(dy)$. In (16), μ_t is the law of the process x_t , and $t \rightarrow \mu'_t$ the derivative of $t \rightarrow \mu_t$ (in the sense of measure theory). Here, \mathcal{L}_{cont}^* is the adjoint of \mathcal{L}_{cont} (the continuous part of the infinitesimal operator of \mathbf{M}) in the sense of distribution theory.

Remark that in the case of stochastic hybrid processes, the forward and backward Kolmogorov equations are *parabolic integro partial differential equations*.

The backward/forward Kolmogorov equations of a stochastic hybrid process that describes the UL-SoS abstraction are based on the expression of the infinitesimal generator (12).

Remark 5 [2] *For spontaneous jumps, a mean jump intensity always exists, and it is the expectation of the transition rate function (stochastic jump intensity) $\lambda(x_t)$ on the event $\{x_t \in \Gamma\}$.*

This is a key remark for our analysis. Then the derivation of the FPK equation seems to be feasible for the ULSoS abstraction process (since it does not exhibit forced transitions). The only problem we encounter is that the expression of the transition rate function (6) for $(\beta_t^i, \Upsilon_t^i)$ is not known! This rate depends on the probability distribution function of the first passage time of the modified active boundary. In the next section, we will exploit additional hypotheses that can make the computation of these jump rates feasible.

6 Conclusions

In this paper, we have proposed a rigorous, mathematical modelling framework for massively parallel multi-agent systems. The purpose of this framework is to allow the top down control. Each agent is a stochastic hybrid system, and the multi-agent system itself is also a stochastic hybrid system. The model has two scales. At the microscopic scale, agents have a coordination dynamics, and an activity dynamics. Each agent can communicate with its neighborhood, a finite set of agents connected via bidirectional communication channels. At the macroscopic level, the dynamics of an ULSoS is modelled as a PDMP. The macroscopic level is useful for composing MPMASs.

The technical contribution consists of determining the expression of the infinitesimal generator of an ULSoS, and the derivation of the associated Kolmogorov equations.

In the future work, we will investigate the topics of logics for specifying and reasoning about massive parallelism, probabilistic model checking of safety and performability properties and identifying multiple scales that can be related by formal refinement/abstraction relations.

To the author knowledge, the approach presented in this paper is new and original. A continuous time continuous space model for swarms has been developed in [7]. Like in this work, there the author considers also a two layers model. There are two major differences compared to this approach. At macroscopic level, the model from [7] is continuous with no possibility of operating regime change. This makes the control more difficult. The second difference comes from the control perspective. In the above reference, the control is bottom-up studying the impact at the macroscopic level of the simple interaction rules from the microscopic level. The approach developed in [3] (and the references therein) is also based on hybrid systems. There, each hybrid agent is deterministic, but the agents interact following the pattern of a chemical reaction network, which is probabilistic. The control is also top-down and two layered. At the macroscopic level, optimization strategies are investigated, while at the microscopic level, the focus is on collision avoidance.

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