# Congruence Closure Modulo Permutation Equations 

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#### Abstract

We present a framework for constructing congruence closure modulo permutation equations, which extends the abstract congruence closure [7] framework for handling permutation function symbols. Our framework also handles certain interpreted function symbols satisfying each of the following properties: idempotency ( $I$ ), nilpotency ( $N$ ), unit $(U), I \cup U$, or $N \cup U$. Moreover, it yields convergent rewrite systems corresponding to ground equations containing permutation function symbols. We show that congruence closure modulo a given finite set of permutation equations can be constructed in polynomial time using equational inference rules, allowing us to provide a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols.


## 1 Introduction

Congruence closure procedures $[12,18,19]$ have been researched for several decades, and play important roles in software/hardware verification (see $[9,19,20]$ ) and satisfiability modulo theories (SMT) solvers $[8,10]$. They provide fast decision procedures for determining whether a ground equation is an (equational) consequence of a given set of ground equations. (The fastest known congruence closure algorithm runs in $O(n \log n)$ [15].)

In [7, 14], some approaches to constructing the congruence closure of ground equations using completion methods were considered. These approaches capture the efficient techniques from standard term rewriting for congruence closure procedures. In particular, the abstract congruence closure approach in [7] (cf. Kapur's approach in [14]) constructs a reduced convergent ground rewrite system $R_{S}$ for a finite set of ground equations $S$, which consists of either rewrite rules of the form $a \rightarrow c$ or $f\left(c_{1}, \ldots, c_{n}\right) \rightarrow c$ or $c \rightarrow d$ for fresh constants $c_{1}, \ldots, c_{n}, c, d$. Furthermore, $R_{S}$ is a conservative extension of the equational theory induced by $S$ (i.e. the congruence closure $C C(S)$ ) on ground terms, and two ground terms are congruent in $C C(S)$ iff they have the same normal form w.r.t. $R_{S}$. Note that this approach does not require a total termination ordering on ground terms.

Congruence closure procedures were extended to congruence closure procedures modulo theories in order to handle interpreted function symbols in the signature [ $3,6,15$ ]. The notion of congruence closure modulo associative and commutative (AC) theories was discussed in [6,16], and the notion of conditional congruence closure with uninterpreted and some interpreted function symbols was considered in [15].

Meanwhile, an equation is a permutation equation [1] if it is of the form $f\left(x_{1}, \ldots, x_{n}\right) \approx f\left(x_{\pi(1)}, \ldots\right.$, $\left.x_{\pi(n)}\right)$, where $\pi$ is a permutation on the set $\{1, \ldots, n\}$. Commutativity is the simplest case of permutation equations. Permutation equations are difficult to handle in equational reasoning without using the modulo approach. For example, an ordered completion procedure for ordered rewriting [5] produces every equation of the form $f\left(x_{1}, x_{2}, \ldots, x_{n}\right) \approx f\left(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}\right)$ (up to variable renaming) from two permutation equations $f\left(x_{1}, x_{2}, \ldots, x_{n}\right) \approx f\left(x_{2}, x_{1}, x_{3}, \ldots, x_{n}\right)$ and $f\left(x_{1}, x_{2}, \ldots, x_{n}\right) \approx f\left(x_{2}, x_{3}, \ldots, x_{n}, x_{1}\right)$, where $\rho$ is a permutation in the symmetric group $S_{n}$ of cardinality $n!$. (Recall that the symmetric group $S_{n}$ can be generated by two cycles (12) and $(12 \cdots n)$.) Therefore, it is natural to view permutation
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equations as "structural axioms" (defining a congruence relation on terms) rather than viewing them as "simplifiers" (defining a reduction relation on terms) [5].

In this paper, we present a framework for generating congruence closure modulo a finite set of permutation equations. To our knowledge, it has not been discussed in the literature, and a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols has not yet been known.

Our framework is based on the notion of abstract congruence closure that is particularly useful for term representation and checking $E$-equality between two flat terms for a given set of permutation equations $E$, which does not require an $E$-compatible ordering (cf. [17]). In addition, it also handles function symbols satisfying each of the following properties: idempotency ( $I$ ), nilpotency ( $N$ ), unit ( $U$ ), $I \cup U$, or $N \cup U$. (If a function symbol is a permutation function symbol satisfying one of the above properties, then it should be a commutative function symbol.)

We show that congruence closure modulo a given finite set of permutation equations (with or without the function symbols satisfying the above properties) can be constructed in polynomial time, which provides a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols (appearing in $E$ ).

## 2 Preliminaries

We use the standard terminology and definitions of term rewriting [4,11], congruence closure [7,12, 19], and permutation groups [13]. We also use some terminology and the results of permutation equations found in [1,2].

We denote by $\mathscr{T}(\mathscr{F}, \mathscr{X})$ the set of terms over a finite set of function symbols $\mathscr{F}$ and a denumerable set of variables $\mathscr{X}$. We denote by $T(\mathscr{F})$ the set of ground terms over $\mathscr{F}$. We assume that each function symbol in $\mathscr{F}$ has a fixed arity.

An equation is an expression $s \approx t$, where $s$ and $t$ are (first-order) terms built from $\mathscr{F}$ and $\mathscr{X}$. A ground equation (resp. ground term) is an equation (resp. a term) which does not contain any variable.

We write $s[u]$ if $u$ is a subterm of $s$ and denote by $s[t]_{p}$ the term that is obtained from $s$ by replacing the subterm at position $p$ of $s$ by $t$.

An equivalence is a reflexive, transitive, and symmetric binary relation. An equivalence $\sim$ on terms is a congruence if $s \sim t$ implies $u[s]_{p} \sim u[t]_{p}$ for all terms $s, t, u$ and positions $p$.

An equational theory is a set of equations. We denote by $\approx_{E}$ (called the equational theory induced by $E$ ) the least congruence on $T(\mathscr{F}, \mathscr{X})$ that is stable under substitutions and contains a set of equations $E$. If $s \approx_{E} t$ for two terms $s$ and $t$, then $s$ and $t$ are $E$-equivalent.

Given a finite set $S=\left\{a_{i} \approx b_{i} \mid 1 \leq i \leq m\right\}$ of ground equations where $a_{i}, b_{i} \in T(\mathscr{F})$, the congruence closure $C C(S)[3,15]$ is the smallest subset of $T(\mathscr{F}) \times T(\mathscr{F})$ that contains $S$ and is closed under the following rules: (i) $S \subseteq C C(S)$, (ii) for every $a \in T(\mathscr{F}), a \approx a \in C C(S)$ (reflexivity), (iii) if $a \approx b \in C C(S)$, then $b \approx a \in C C(S)$ (symmetry), (iv) if $a \approx b$ and $b \approx c \in C C(S)$, then $a \approx c \in C C(S)$ (transitivity), and (v) if $f \in \mathscr{F}$ is an $n$-ary function symbol $(n>0)$ and $a_{1} \approx b_{1}, \ldots, a_{n} \approx b_{n} \in C C(S)$, then $f\left(a_{1}, \ldots, a_{n}\right) \approx$ $f\left(b_{1}, \ldots, b_{n}\right) \in C C(S)$ (monotonicity). Note that $C C(S)$ is also the equational theory induced by $S$.

A (strict) ordering $\succ$ on terms is an irreflexive and transitive relation on $T(\mathscr{F}, \mathscr{X})$.
Given a rewrite system $R$ and a set of equations $E$, the rewrite relation $\rightarrow_{R, E}$ on $T(\mathscr{F}, \mathscr{X})$ is defined by $s \rightarrow_{R, E} t$ if there is a non-variable position $p$ in $s$, a rewrite rule $l \rightarrow r \in R$, and a substitution $\sigma$ such that $\left.s\right|_{p} \approx_{E} l \sigma$ and $t=s[r \sigma]_{p}$. The transitive and reflexive closure of $\rightarrow_{R, E}$ is denoted by ${ }^{*}{ }_{R, E}$. We say that a term $t$ is an $R, E$-normal form if there is no term $t^{\prime}$ such that $t \rightarrow_{R, E} t^{\prime}$.

The rewrite relation $\rightarrow_{R / E}$ on $T(\mathscr{F}, \mathscr{X})$ is defined by $s \rightarrow_{R / E} t$ if there are terms $u$ and $v$ such that $s \approx_{E} u, u \rightarrow_{R} v$, and $v \approx_{E} t$. We simply say the rewrite relation $\rightarrow_{R / E}$ (resp. $\rightarrow_{R, E}$ ) on $T(\mathscr{F}, \mathscr{X})$ as the rewrite relation $R / E$ (resp. $R, E$ ).

The rewrite relation $R, E$ is Church-Rosser modulo $E$ if for all terms $s$ and $t$ with $s \stackrel{*}{\hookrightarrow}_{R \cup E} t$, there are terms $u$ and $v$ such that $s \stackrel{*}{\rightarrow}_{R, E} u \stackrel{*}{{ }^{*}} E v \stackrel{*}{\leftarrow} R, E$. The rewrite relation $R, E$ is convergent modulo $E$ if $R, E$ is Church-Rosser modulo $E$ and $R / E$ is well-founded.

The depth of a term $t$ is defined as $\operatorname{depth}(t)=0$ if $t$ is a variable or a constant and $\operatorname{depth}\left(f\left(s_{1}, \ldots, s_{n}\right)\right)$ $=1+\max \left\{\operatorname{depth}\left(s_{i}\right) \mid 1 \leq i \leq n\right\}$. A term $t$ is flat if its depth is 0 or 1 .

An equation of the form $f\left(x_{1}, \ldots, x_{n}\right)=f\left(x_{\rho(1)}, \ldots, x_{\rho(n)}\right)$ is a permutation equation [1] if $\rho$ is a permutation on $\{1, \ldots, n\}$. We use variable naming in such a way that the left-hand side of each equation in a set of permutation equations with the same function symbol has the same name of variables $x_{1}, \ldots, x_{k}$ from left to right. (In this paper, we assume that the set of function symbols $\mathscr{F}$ in $T(\mathscr{F}, \mathscr{X})$ is finite and each function symbol in $\mathscr{F}$ has a fixed arity.)

We denote by $\mathscr{F}_{E}$ the set of all function symbols occurring in a finite set of permutation equations $E$.
If $e:=f\left(x_{1}, \ldots, x_{n}\right) \approx f\left(x_{\rho(1)}, \ldots, x_{\rho(n)}\right)$ is a permutation equation, then $\rho$ is the permutation of this equation. We denote by $\pi[e]$ the permutation of $e$. For example, $\rho$ is the permutation of the permutation equation $e^{\prime}:=f\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \approx f\left(x_{1}, x_{3}, x_{2}, x_{4}\right)$ (i.e. $\pi\left[e^{\prime}\right]=\rho$ ) with $\rho(1)=1, \rho(2)=3, \rho(3)=2$, and $\rho(4)=4$. Let $E$ be a set of permutation equations with the same top function symbol. Then $\Pi[E]$ is defined as $\Pi[E]:=\{\pi[e] \mid e \in E\}$. The permutation group generated by $\Pi[E]$ is denoted by $<\Pi[E]>$.
Theorem 1. (see Theorem 1.4 in [2]) Let E be a set of permutation equations and let $e$ be a permutation equation such that every equation in $E \cup\{e\}$ has the same (top) function symbol. Then $E \models e$ if and only if $\pi[e] \in\langle\Pi[E]>$.

Let $i_{1}, i_{2}, \ldots, i_{r}(r \leq n)$ be distinct elements of $I_{n}=\{1,2, \ldots, n\}$. Then $\left(i_{1} i_{2} \cdots i_{r}\right)$, called a cycle of length $r$, is defined as the permutation that maps $i_{1} \mapsto i_{2}, i_{2} \mapsto i_{3}, \ldots, i_{r-1} \mapsto i_{r}$ and $i_{r} \mapsto i_{1}$, and every other element of $I_{n}$ maps onto itself. The symmetric group $S_{n}$ of cardinality $n$ ! can be generated by two cycles (12) and ( $12 \cdots n$ ).
Example 1. Let $E=\left\{f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right) \approx f\left(x_{2}, x_{1}, x_{3}, x_{4}, x_{5}\right), f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right) \approx f\left(x_{2}, x_{3}, x_{4}, x_{5}, x_{1}\right)\right\}$. Then $\Pi[E]$ consists of two cycles $\{(12),(12345)\}$. Since two cycles (12) and (12345) generate the symmetric group $S_{5}$, we see that $\left\langle\Pi[E]>\right.$ is $S_{5}$. Therefore, $f\left(x_{1}, \ldots, x_{5}\right) \approx_{E} f\left(x_{\tau(1)}, \ldots, x_{\tau(5)}\right)$ for any permutation $\tau \in S_{5}$ by Theorem 1 .

Let $E$ be a finite set of permutation equations. Then $E$ can be uniquely decomposed as $\bigcup_{i=1}^{n} E_{i}$ such that (i) each $E_{i}$ is a finite set of permutation equations, and (ii) $E_{j}$ and $E_{k}$ with $j \neq k$ are disjoint such that if $s_{j} \approx t_{j} \in E_{j}$ and $s_{k} \approx t_{k} \in E_{k}$, then $s_{j}$ and $s_{k}$ do not have the same top symbol (and are not variants of each other). Since we assume that each function symbol has a fixed arity, each distinct function symbol occurring in $E$ corresponds to a distinct $E_{i}$ in $E$. We denote by $E q(f)$ the corresponding equational theory with terms headed by such a function symbol $f$. Now, we may apply Theorem 1 for each $E q(f)$ in $E$ with $f \in \mathscr{F}_{E}$.

## 3 Congruence closure modulo permutation equations

Definition 2. Let $K$ be a set of constants disjoint from $\mathscr{F}$.
(i) A $D$-rule (w.r.t. $\mathscr{F}$ and $K$ ) is a rewrite rule of the form $f\left(c_{1}, \ldots, c_{n}\right) \rightarrow c$, where $c_{1}, \ldots, c_{n}, c$ are constants in $K$ and $f \in \mathscr{F}$ is an $n$-ary function symbol.
(ii) A $C$-rule (w.r.t. $K$ ) is a rule $c \rightarrow d$, where $c$ and $d$ are constants in $K$.

In Definition 2(i), note that $f \in \mathscr{F}$ can also be a 0 -ary function symbol (i.e. a constant).
Example 2. Let $E=\left\{f\left(x_{1}, x_{2}\right) \approx f\left(x_{2}, x_{1}\right), g\left(x_{1}, x_{2}, x_{3}\right) \approx g\left(x_{2}, x_{1}, x_{3}\right)\right\}$. If $\mathscr{F}=\{a, b, h, f, g\}$ with $\mathscr{F}_{E}=\{f, g\}$ and $P=\{f(b, g(b, a, a)) \approx h(a)\}$, then $D_{0}=\left\{a \rightarrow c_{0}, b \rightarrow c_{1}, g\left(c_{1}, c_{0}, c_{0}\right) \rightarrow c_{2}, f\left(c_{1}, c_{2}\right) \rightarrow\right.$ $\left.c_{3}, h\left(c_{0}\right) \rightarrow c_{4}\right\}$ is a possible set of $D$-rules over $\mathscr{F}$, and we have $K=\left\{c_{0}, c_{1}, c_{2}, c_{3}, c_{4}\right\}$. Using $D_{0}$, we can simplify the original equations in $P$, which gives the set of $C$ rules, i.e., $C_{0}=\left\{c_{3} \rightarrow c_{4}\right\}$ where $c_{3} \succ c_{4}$.
Definition 3. Let $E$ be a finite set of permutation equations and $K$ be a set of constants disjoint from $\mathscr{F}$. A ground rewrite system $R=D \cup C$ is a congruence closure modulo $E$ (w.r.t. $\mathscr{F}$ and $K$ ) if the following conditions are met:
(i) $D$ is a set of $D$-rules and $C$ is a set of $C$-rules, and for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\hookrightarrow}_{R, E} c$.
(ii) $R, E$ is a ground convergent (modulo $E$ ) rewrite system over $\mathscr{T}(\mathscr{F} \cup K)$.

In addition, given a set of ground equations $P$ over $\mathscr{T}(\mathscr{F} \cup K), R$ is said to be a congruence closure modulo $E$ (w.r.t. $\mathscr{F}$ and $K$ ) for $P$ if for all ground terms $s$ and $t$ over $\mathscr{T}(\mathscr{F}), s \stackrel{*}{\longleftrightarrow} P \cup E t$ iff there are ground terms $u$ and $v$ over $\mathscr{T}(\mathscr{F} \cup K)$ such that $s \stackrel{*}{\rightarrow}_{R, E} u \stackrel{*^{*}}{E} v \stackrel{*}{\leftarrow}_{R, E} t$.

In the following, by $B$-rules with the interpreted function symbol $g \in \mathscr{F}$, we mean either the idempotency rule (I): $\{g(x, x) \rightarrow x\}$ or the nilpotency rule $(N):\{g(x, x) \rightarrow 0\}$ or the unit rule $(U):\{g(x, 0) \rightarrow$ $x, g(0, x) \rightarrow x\}$ or $I \cup U$ or $N \cup U$.

Definition 4. Let $E$ be a finite set of permutation equations and $K$ be a set of constants disjoint from $\mathscr{F}$. A ground rewrite system $R=D \cup C$ is a congruence closure modulo $E \cup B$ (w.r.t. $\mathscr{F}$ and $K$ ) if the following conditions are met:
(i) $B$ is a set of $B$-rules with the interpreted function symbol $g \in \mathscr{F} .{ }^{1}$
(ii) $D$ is a set of $D$-rules and $C$ is a set of $C$-rules, and for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\hookrightarrow}_{R, E} c$.
(iii) $R \cup B, E$ is a convergent (modulo $E$ ) rewrite system over $\mathscr{T}(\mathscr{F} \cup K, \mathscr{X}) .{ }^{2}$

In addition, given a set of ground equations $P$ over $\mathscr{T}(\mathscr{F} \cup K), R$ is said to be a congruence closure modulo $E \cup B$ (w.r.t. $\mathscr{F}$ and $K$ ) for $P$ if for all ground terms $s$ and $t$ over $\mathscr{T}(\mathscr{F}), s \stackrel{*}{\hookrightarrow}_{P \cup B \cup E} t$ iff there are ground terms $u$ and $v$ over $\mathscr{T}(\mathscr{F} \cup K)$ such that $s \xrightarrow{*}_{R \cup B, E} u \stackrel{*}{\leftrightarrow}_{E} v \stackrel{*}{*}_{R \cup B, E} t$.

Note that $B$ or $E$ can be empty in Definition 4. If $B$ is empty, then it is the same as Definition 3. Also, condition (ii) in Definition 4 states that each constant $c$ in $K$ represents some term in $\mathscr{T}(\mathscr{F})$ w.r.t. $R, E$, meaning that $K$ contains no superfluous constants (cf. [7]).

Definition 5. We denote by $W$ the infinite set of constants $\left\{c_{0}, c_{1}, \ldots\right\}$ such that $W$ is disjoint from $\mathscr{F}$, and denote by $K$ a finite subset chosen from $W$. We define orderings $\succ_{K}$ on $K$, and $\succ$ and $\succ_{l p o}$ on $\mathscr{T}(\mathscr{F} \cup K)$ as follows:
(i) $c_{i} \succ_{K} c_{j}$ if $i<j$ for all $c_{i}, c_{j} \in K$.
(ii) $c \succ d$ if $c \succ_{K} d$, and $t \succ c$ if $t \rightarrow c$ is a $D$-rule.

[^0](iii) $\succ_{l p o}$ is a lexicographic path ordering on $\mathscr{T}(\mathscr{F} \cup K)$, which can be defined from the following assumptions:
(iii.1) $c \succ_{l p o} d$ if $c \succ_{K} d$,
(iii.2) $t \succ_{l p o} c$ if $t$ is any term headed by a function symbol $f$ in $\mathscr{F}$ and $c$ is any constant in $K$, and
(iii.3) there is a total precedence on symbols in $\mathscr{F}$.

Observe that $\succ_{l p o}$ extends $\succ$, and is total on $\mathscr{T}(\mathscr{F} \cup K)$. (If the precedence on $\mathscr{F} \cup K$ is total, then the associated lexicographic path ordering $\succ_{l p o}$ is total on $\mathscr{T}(\mathscr{F} \cup K)$ (see [11]).) We emphasize that a partial ordering $\succ$ on $\mathscr{T}(\mathscr{F} \cup K)$ suffices for inference rules in Figure 1.

Figure 1 shows the inference rules for congruence closure modulo permutation equations, which extends the inference rules for the abstract congruence closure framework in [7]. We have the additional inference rule called the REWRITE rule in Figure 1 . Also, we use the $E$-equality $\approx_{E}$ instead of the equality $\approx$ for the DEDUCE and DELETE inference rules. We write $(K, P, R) \vdash\left(K^{\prime}, P^{\prime}, R^{\prime}\right)$ to indicate that ( $K^{\prime}, P^{\prime}, R^{\prime}$ ) can be obtained from ( $K, P, R$ ) by application of an inference rule in Figure 1, where $K$ denotes a set of new constants (see Definition 5), $P$ a set of equations, and $R$ a set of rewrite rules consisting of $C$-rules and $D$-rules. Also, in Figure $1, B$ denotes a set of $B$-rules. A derivation is a sequence of states $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$.
Lemma 6. If $(K, P, R) \vdash\left(K^{\prime}, P^{\prime}, R^{\prime}\right)$, then for all $u$ and $v$ in $\mathscr{T}(\mathscr{F} \cup K)$, we have $u \stackrel{*}{\leftrightarrow} E \cup B \cup P^{\prime} \cup R^{\prime} v$ if and only if $u \stackrel{*}{\hookrightarrow}_{E \cup B \cup P \cup R} v$.

Proof. We consider each application of an inference rule $\tau$ for $(K, P, R) \vdash\left(K^{\prime}, P^{\prime}, R^{\prime}\right)$. If $\tau$ is EXTEND, SIMPLIFY, ORIENT, DELETE, COLLAPSE, or COMPOSE, then the conclusion can be verified similarly to [5,7].

If $\tau$ is REWRITE, then we let $P=\bar{P}, R=\bar{R} \cup\left\{l^{\prime} \rightarrow r^{\prime}\right\}, R^{\prime}=\bar{R}, P^{\prime}=\bar{P} \cup\left\{r \sigma \approx r^{\prime}\right\}$, and $K=K^{\prime}$. Since $(K \cup P \cup R)-\left(K^{\prime} \cup P^{\prime} \cup R^{\prime}\right)=\left\{l^{\prime} \rightarrow r^{\prime}\right\}$, we need to show that $l^{\prime} \stackrel{*}{\leftrightarrow}{ }_{E \cup B \cup P^{\prime} \cup R^{\prime} r^{\prime} \text {. As } l^{\prime}=l \sigma \rightarrow_{B} r \sigma \leftrightarrow P^{\prime} r^{\prime}, ~ ; ~, ~}^{\text {, }}$ we have $l^{\prime} \stackrel{*}{\leftrightarrow}_{E \cup B \cup P^{\prime} \cup R^{\prime}} r^{\prime}$. Conversely, since $\left(K^{\prime} \cup P^{\prime} \cup R^{\prime}\right)-(K \cup P \cup R)=\left\{r \sigma \approx r^{\prime}\right\}$, we need to show that $r \sigma \stackrel{*}{\leftrightarrows}{ }_{E \cup B \cup P \cup R} r^{\prime}$. As $r \sigma \leftarrow_{B} l \sigma=l^{\prime} \rightarrow_{R} r^{\prime}$, we have $r \sigma \stackrel{*}{\hookrightarrow}_{E \cup B \cup P \cup R} r^{\prime}$.

If $\tau$ is DEDUCE, then let $R=\bar{R} \cup\{s \rightarrow c, t \rightarrow d\}, P^{\prime}=P \cup\{c \approx d\}, R^{\prime}=\bar{R} \cup\{t \rightarrow d\}$, and $K=K^{\prime}$, where $s \approx_{E} t$. Since $(K \cup P \cup R)-\left(K^{\prime} \cup P^{\prime} \cup R^{\prime}\right)=\{s \rightarrow c\}$, we need to show that $s \stackrel{*}{\hookrightarrow}_{E \cup B \cup P^{\prime} \cup R^{\prime}} c$. As $s \stackrel{*}{\hookrightarrow}_{E} t \rightarrow_{R^{\prime}} d \leftrightarrow_{P^{\prime}} c$, we have $s \stackrel{*}{\leftrightarrow}_{E \cup B \cup P^{\prime} \cup R^{\prime}} c$. Conversely, since $\left(K^{\prime} \cup P^{\prime} \cup R^{\prime}\right)-(K \cup P \cup R)=\{c \approx d\}$, we need to show that $c \stackrel{*}{\leftrightarrows}_{E \cup B \cup P \cup R} d$. As $c \leftarrow_{R} s \stackrel{*}{\leftrightarrow}_{E} t \rightarrow_{R} d$, we have $c \stackrel{*}{\leftrightarrow} E \cup B \cup P \cup R d$.

Definition 7. (i) A derivation is said to be fair if any inference rule that is continuously enabled is applied eventually.
(ii) By a fair $\mu$-derivation, we mean that the EXTEND and SIMPLIFY rule are applied eagerly in a fair derivation.

Theorem 8. Let $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$ be a fair $\mu$-derivation such that $P_{0}$ is a finite set of ground equations with $K_{0}=\emptyset$ and $R_{0}=\emptyset$.
(i) Each fair $\mu$-derivation starting from the initial state $\left(K_{0}, P_{0}, R_{0}\right)$ is finite.
(ii) If $\left(K_{n}, P_{n}, R_{n}\right)$ is a final state (i.e. no inference rule can be applied to $\left(K_{n}, P_{n}, R_{n}\right)$ ) of a fair $\mu$ derivation starting from the initial state $\left(K_{0}, P_{0}, R_{0}\right)$, then $R_{n} \cup B, E$ is convergent modulo $E$, and $R_{n}$ is a congruence closure modulo $E \cup B$ for $P_{0}$.

Proof. Since $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$ is a fair $\mu$-derivation, this derivation can be written as $\left(K_{0}, P_{0}, R_{0}\right) \vdash^{*}\left(K_{m}, P_{m}, R_{m}\right) \vdash\left(K_{m+1}, E_{m+1}, R_{m+1}\right) \vdash \cdots$, where the derivation $\left(K_{m}, P_{m}, R_{m}\right) \vdash\left(K_{m+1}, E_{m+1}\right.$, $\left.R_{m+1}\right) \vdash \cdots$ does not involve any application of the EXTEND rule, so we have the set $K_{m}=K_{m+1}=\cdots$.

EXTEND: $\frac{(K, P[t], R)}{(K \cup\{c\}, P[c], R \cup\{t \rightarrow c\})}$
if $t \rightarrow c$ is a $D$-rule, $c \in W-K$, and $t$ occurs in some equation in $P$.

SIMPLIFY: $\frac{(K, P[t], R \cup\{t \rightarrow c\})}{(K, P[c], R \cup\{t \rightarrow c\})}$
if $t$ occurs in some equation in $P .^{3}$
REWRITE: $\frac{\left(K, P, R \cup\left\{l^{\prime} \rightarrow r^{\prime}\right\}\right)}{\left(K, P \cup\left\{r \sigma \approx r^{\prime}\right\}, R\right)}$
if $l^{\prime}=l \sigma$, where $l \rightarrow r \in B$.
ORIENT: $\quad \frac{(K, P \cup\{s \approx t\}, R)}{(K, P, R \cup\{s \rightarrow t\})}$
if $s \succ t$, and $s \rightarrow t$ is a $D$-rule or a $C$-rule.
DEDUCE: $\quad \frac{(K, P, R \cup\{s \rightarrow c, t \rightarrow d\})}{(K, P \cup\{c \approx d\}, R \cup\{t \rightarrow d\})}$
if $s \approx_{E} t$.
DELETE: $\quad \frac{(K, P \cup\{s \approx t\}, R)}{(K, P, R)}$
if $s \approx_{E} t$.
COMPOSE: $\frac{(K, P, R \cup\{t \rightarrow c, c \rightarrow d\})}{(K, P, R \cup\{t \rightarrow d, c \rightarrow d\})}$
COLLAPSE: $\frac{\left(K, P, R \cup\left\{t[c] \rightarrow c^{\prime}, c \rightarrow d\right\}\right)}{\left(K, P, R \cup\left\{t[d] \rightarrow c^{\prime}, c \rightarrow d\right\}\right)}$
if $c$ is a proper subterm of $t$ and $c \rightarrow d$ is a $C$-rule.

Figure 1: Inference rules for congruence closure modulo permutation equations

For (i), we provide a more concrete result in the following Lemma 9.
For (ii), let ( $K_{n}, P_{n}, R_{n}$ ) be a final state of a fair $\mu$-derivation starting from the state ( $K_{0}, P_{0}, R_{0}$ ). (Note that each fair $\mu$-derivation starting from the initial state ( $K_{0}, P_{0}, R_{0}$ ) is finite by (i), so we have some final state.) Observe that $P_{m}, P_{m+1}, \ldots$ either contains only $C$-equations or is empty, and $\succ$ can orient those $C$ equations, so $P_{n}=\emptyset$. Since $l \succ_{l p o} r$ for all rules $l \rightarrow r \in R_{n} \cup B$, we see that $R_{n} \cup B / E$ is terminating.

Also, since $R_{n} \cup B$ is non-overlapping w.r.t. the rewrite system $R_{n} \cup B, E$ (i.e. there is no non-trivial critical pair between rules in $\left.R_{n} \cup B\right), R_{n} \cup B, E$ is Church-Rosser modulo $E$ by the critical pair lemma [5]. Thus, $R_{n} \cup B, E$ is convergent modulo $E$.

Finally, we show that for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\leftrightarrow} R_{n}, E$ c by induction. Let $c$ be a constant in $K$ and $f\left(c_{1}, \ldots, c_{k}\right) \rightarrow c$ be the corresponding extension rule for $c$ when $c$ was added. By induction hypothesis, we have $s_{i} \stackrel{*}{\leftrightarrow} R_{n}, E$ c $c_{i}$, and thus $\left.f\left(s_{1}, \ldots, s_{k}\right) \stackrel{*}{\leftrightarrow} R_{n}, E\right) f\left(c_{1}, \ldots, c_{k}\right) \rightarrow_{\cup_{i} R_{i}} c$. By Lemma 6 , we also have $f\left(s_{1}, \ldots, s_{k}\right) \stackrel{*}{\leftrightarrow}_{R_{n} \cup P_{n} B \cup E} c$, and thus $f\left(s_{1}, \ldots, s_{k}\right) \stackrel{*}{{ }_{R}} R_{n} \cup B \cup E$ b because $P_{n}=\emptyset$. As $R_{n} \cup B, E$ is convergent modulo $E$ and no more REWRITE rule can be applied to $f\left(s_{1}, \ldots, s_{k}\right)$ by fairness of the derivation, we have $f\left(s_{1}, \ldots, s_{k}\right) \stackrel{*}{\leftrightarrow} R_{n}, E$. Thus $R_{n}$ is a congruence closure modulo $E \cup B$ for $P_{0}$.

In the following lemma, recall that function symbols in $\mathscr{F}$ include 0 -ary function symbols in $\mathscr{F}$, i.e., constants in $\mathscr{F}$.
Lemma 9. Let $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$ be a fair $\mu$-derivation such that $P_{0}$ is a finite set of ground equations with $K_{0}=\emptyset$ and $R_{0}=\emptyset$. Then its derivation length is bounded by $O\left(n^{2}\right)$, where $n$ is the sum of the sizes (number of symbols) of the left-hand and right-hand sides of equations in $P_{0}$.

Proof. We show that the number of applications of each rule in Figure 1 in a fair $\mu$-derivation is bounded above by $O\left(n^{2}\right)$, where $n$ is the sum of the sizes (number of symbols in $\mathscr{F}$ ) of the left-hand and righthand sides of equations in $P_{0}$. Since $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$ is a fair $\mu$-derivation, we may write this derivation as

$$
\left(K_{0}, P_{0}, R_{0}\right) \vdash^{*}\left(K_{m}, P_{m}, R_{m}\right) \vdash\left(K_{m+1}, E_{m+1}, R_{m+1}\right) \vdash \cdots,
$$

where the derivation $\left(K_{m}, P_{m}, R_{m}\right) \vdash\left(K_{m+1}, E_{m+1}, R_{m+1}\right) \vdash \cdots$ does not involve any application of the EXTEND rule, and thus we have the finite set $K_{m}=K_{m+1}=\cdots$.
(i) The total number of the EXTEND inference steps is bounded by $O(n)$. This is because the total number of $\mathscr{F}$-symbols in the second component of the state is not increasing for each transition step ${ }^{4}$ and each EXTEND inference step decreases this number by one.
(ii) A derivation step by the SIMPLIFY, REWRITE, DEDUCE, COMPOSE, or COLLAPSE rule either reduces the number of function symbols of $\mathscr{F}$ in $R_{i} \cup P_{i}$ or rewrites some constant. The length of a rewriting sequence $c_{1} \rightarrow c_{2} \rightarrow \cdots$ is bounded by $\left|K_{m}\right|$. (Here, $\left|K_{m}\right|$ is $O(n)$ because the total number of the EXTEND inference steps is bounded by $O(n)$ as discussed in (i).) Also, the total number of symbols in $P_{i} \cup R_{i}$ is bounded by $O\left(n+\left|K_{m}\right|\right),{ }^{5}$ which is also $O(n)$. This means that the total number of the SIMPLIFY, DEDUCE, COLLAPSE, or COMPOSE inference steps is bounded

[^1]by $O\left(n^{2}\right)$. (Note that rewriting constants takes $O\left(n^{2}\right)$ because there are at most $O(n)$ constants, and the length of a rewriting sequence for each constant is bounded by $O(n)$.)
(iii) The total number of the DELETE inference steps is bounded by $O\left(n+\left|K_{m}\right|\right)$ (i.e. $O(n)$ ) because the total number of symbols in $P_{i} \cup R_{i}$ is bounded by $O\left(n+\left|K_{m}\right|\right)$.
(iv) The total number of the ORIENT inference steps is bounded by the total number of EXTEND, SIMPLIFY, DEDUCE, COLLAPSE, and COMPOSE inference steps, which is $O\left(n^{2}\right)$. Note that each ORIENT inference step neither increases the number of function symbols nor the number of constants.

Thus, the derivation length of any fair $\mu$-derivation starting from $\left(K_{0}, P_{0}, R_{0}\right)$ is bounded by $O\left(n^{2}\right)$.
Given a finite (fixed) set of permutation equations $E$ and two terms $s=f\left(s_{1}, \ldots, s_{k}\right)$ and $t=f\left(t_{1}, \ldots\right.$, $t_{k}$ ) with $f \in \mathscr{F}_{E}$, we can determine whether $s \approx_{E} t$ in $O\left(n^{2}\right)$ time (measured in $\left.n=|s|+|t|\right)$ using an additional data structure (i.e. a table) that can be constructed in polynomial time [1]. If $s$ and $t$ are both flat, then we can determine whether $s \approx_{E} t$ in $O(n)$ time using the following procedure with a table that can be constructed in polynomial time (see [1]).

## Equality-Test( $s, t$ )

Input: $s=f\left(c_{1}, \ldots, c_{i}\right)$ and $t=g\left(d_{1}, \ldots, d_{j}\right)$, where $s$ and $t$ are both flat.
Output: If $s \approx_{E} t$, then return true. Otherwise, return false.

1. Determine whether $s$ and $t$ are headed by the same function symbol (i.e. $f=g$ and thus $i=j$ ). If not, then return false. If it is true, then consider the following:
2. Determine whether $f \in \mathscr{F}_{E}$. If not, then $s$ and $t$ are compared by syntactic equality, and return true if they are syntactically equal. Otherwise, if $f \in \mathscr{F}_{E}$, then consider the following:
3. Determine whether $s \approx_{E} t$ using the TestEq procedure in [1].

It is easy to see that steps 1 and 2 of the Equality-Test $(s, t)$ procedure take at most $O(n)$ time. For step 3, which corresponds to the case $f=g$ and $f \in \mathscr{F}_{E}$, it takes $O(n)$ time for comparing two multisets.

If they are equal, then $s$ and $t$ are further compared in constant time using the Test $E q$ procedure in [1] with a table that can be constructed in polynomial time. (Note that the arity of all $f \in \mathscr{F}_{E}$ and the size of the data structure (i.e. table) is bounded by a constant independent of the size of the input terms.) Therefore, the Equality-Test $(s, t)$ procedure takes $O(n)$ time using a table that can be constructed in polynomial time. In what follows, we denote by $\operatorname{Table}(E q(f))$ this table for each $f \in \mathscr{F}_{E}$ for a finite (fixed) set of permutation equations $E$.

Theorem 10. Given the table Table $(E q(f))$ for each $f \in \mathscr{F}_{E}$, a congruence closure modulo $E \cup B$ for a finite set of ground equations $P$ can be computed in $O\left(n^{3}\right)$ time, where $n$ is the sum of the sizes (number of symbols) of the left and right sides of equations in $P$.

Proof. We first construct a fair $\mu$-derivation $\left(K_{0}, P_{0}, R_{0}\right) \vdash\left(K_{1}, P_{1}, R_{1}\right) \vdash \cdots$ such that $P_{0}$ is a finite set of ground equations with $K_{0}=\emptyset$ and $R_{0}=\emptyset$.

It is easy to see that each EXTEND, SIMPLIFY, ORIENT, COMPOSE, and COLLAPSE inference step in the derivation takes $O(n)$ time.

Each REWRITE inference step in the derivation takes $O(n)$ time because we only need to consider for rules $g(x, x) \rightarrow x, g(x, x) \rightarrow 0, g(x, 0) \rightarrow x$, and $g(0, x) \rightarrow x$ for some interpreted function symbol $g \in \mathscr{F}$.

Each DEDUCE inference step in the derivation takes $O(n)$ time for checking $E$-equality (see the

Equality-Test( $s, t$ ) procedure) between two left-hand side terms $s$ and $t$ in $R_{i}$. Similarly, each DELETE inference step in the derivation takes $O(n)$ time for checking $E$-equality.

By Lemma 9, we know that the derivation length of a fair $\mu$-derivation is bounded by $O\left(n^{2}\right)$. Since each inference step in the derivation takes $O(n)$ time, a congruence closure modulo $E \cup B$ for a finite set of ground equations $P$ can be computed in $O\left(n^{3}\right)$ time.

Corollary 11. The word problem for a finite set of ground equations $P$ with a fixed set of permutation function symbols is decidable in polynomial time.

Proof. We can decide whether $s \approx_{E}^{?} t$ for two ground terms $s$ and $t$ using a congruence closure modulo $E$ for $P$. By Theorem 10, we can compute a congruence closure modulo $E$ for $P$ in polynomial time by constructing and using the table $\operatorname{Table}(E q(f))$ for each $f \in \mathscr{F}_{E}$. Let $R$ be a congruence closure modulo $E$ for $P$. We obtain each normal form of $s$ and $t$ using $R$. We first rewrite each constant symbol in $\mathscr{F}$ of $s$ and $t$ to a new constant symbol in $K$ obtained from constructing $R$, which takes $O(m)$ time where $m=|s|+|t|$. Each rewrite step either reduces the size of a term or rewrites a constant in $K$ to another constant in $K$. The length of a rewriting sequence $c_{1} \rightarrow c_{2} \rightarrow \cdots$ is bounded by $|K|$ (i.e. $O(n)$ ), where $n$ is the sum of the sizes of the left-hand and right-hand sides of equations in $P$. We may also infer that the sum of the sizes of the left-hand and right-hand sides of the rewrite rules in $R$ is $O(n+|K|)$, which is $O(n)$. Each rewrite step takes at most $O\left(n^{2}\right)$ time using $R$ and the Equality-Test procedure. By combining these steps together, we can decide whether $s \approx_{E}^{?} t$ for two ground terms $s$ and $t$ using their normal forms in polynomial time.

The above corollary also holds if some function symbols (not necessarily permutation function symbols) satisfies the properties, such as idempotency ( $I$ ), nilpotency ( $N$ ), unit ( $U$ ), $I \cup U$, or $N \cup U$.

## 4 Example of congruence closure modulo $E \cup B$

Let $B$ be the set of the equation for an idempotency function symbol $g$, i.e., $B=\{g(x, x) \rightarrow x\}$ and let $E$ be the following set of permutation equations:

$$
\begin{aligned}
E=\{ & f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx f\left(x_{2}, x_{1}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right), \\
& f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx f\left(x_{2}, x_{3}, x_{4}, x_{1}, x_{5}, x_{6}, x_{7}, x_{8}\right), \\
& f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{6}, x_{5}, x_{7}, x_{8}\right), \\
& \left.f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{8}, x_{7}\right)\right\} .
\end{aligned}
$$

In this example, we may view each variable $x_{i}$ as a switch in a specially designed electric board, where each variable will be assigned to either constant $T$ (representing "on") or constant $F$ (representing "off"). Each ground term $f\left(c_{1}, c_{2}, c_{3}, c_{4}, c_{5}, c_{6}, c_{7}, c_{8}\right)$ with $c_{i}=T$ or $F$ represents a certain state of this electric board. There is a special transformation button in this electric board, which may transform one state to another state of the electic board. This transformation button is represented by a function with symbol $h \notin \mathscr{F}_{E}$. The problem is to determine if a certain state in the electric board (represented by a term) generates a fault state (represented by term $\perp$ ). We see that $\Pi[E]=\{(12),(1234),(56),(78)\}$, which means that $f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx_{E} f\left(x_{\rho(1)}, x_{\rho(2)}, x_{\rho(3)}, x_{\rho(4)}, x_{5}, x_{6}, x_{7}, x_{8}\right)$ for any permutation $\rho$ on the set $\{1,2,3,4\}, f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx_{E} f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{\pi(5)}, x_{\pi(6)}, x_{7}, x_{8}\right)$ for any permutation $\pi$ on the set $\{5,6\}$, and $f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \approx_{E} f\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{\tau(7)}, x_{\tau(8)}\right)$ for any permutation $\tau$
on the set $\{7,8\}$ (see Thereom 1). Therefore, eight switches in the board are partitioned into three components, i.e. $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\},\left\{x_{5}, x_{6}\right\}$ and $\left\{x_{7}, x_{8}\right\}$, where the order of "switch on" or "switch off" does not matter in each component. For example, $f(T, T, F, F, T, F, T, F) \approx_{E} f(F, F, T, T, F, T, T, F)$. Meanwhile, $g$ is an idempotent function symbol, which serves as a comparator for fault states. For example, if $g(\perp, f(F, F, F, T, T, T, T, F))$, then it is $\perp$ if $f(F, F, F, T, T, T, T, F)$ is $\perp$. Now we start with the following set of ground equations:

1. $f(T, T, T, T, T, T, T, T) \approx \perp$
2. $h(f(F, F, F, F, F, F, F, F)) \approx f(F, T, F, T, F, T, F, T)$
3. $f(T, F, F, F, F, F, F, T) \approx g(\perp, h(f(T, T, T, T, F, T, F, T)))$
4. $h(f(T, F, T, F, T, F, T, F)) \approx f(F, F, F, F, T, T, T, T)$
5. $f(F, F, F, F, T, T, T, T) \approx f(T, T, T, T, F, F, F, F)$
6. $h(f(T, T, T, T, F, F, F, F)) \approx f(T, T, T, T, T, F, T, F)$
7. $h(f(T, T, T, T, F, T, F, T)) \approx f(T, T, T, T, T, T, T, T)$

We show that, for example, each of $h^{4}(f(F, F, F, F, F, F, F, F))$ and $f(T, F, F, F, F, F, F, T)$ is a fault state. (For notational brevity, by $h^{i}(t)$, we mean the function symbol $h$ is applied to term $h^{i-1}(t)$ with $h^{0}(t)$ denoting $t$.) The initial state is ( $K_{0}, P_{0}, R_{0}$ ), where $K_{0}=R_{0}=\emptyset$ and $P_{0}$ consists of the above equations $1-7$. We apply a fair $\mu$-derivation starting with ( $K_{0}, P_{0}, R_{0}$ ) and some intermediate and repetitive steps are omitted for clarity. In the following, each rewrite rule is an element of some $R_{i}$ and each equation is an element of some $P_{j}$. We assume that $c_{i} \succ c_{j}$ if $i<j$.

1(a). $\quad T \rightarrow c_{1}, F \rightarrow c_{2}, \perp \rightarrow c_{3}$
EXTEND and SIMPLIFY for 1
1(b). $\quad f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{4}$
$1(c) . \quad c_{4} \approx c_{3}$
2(a). $f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}\right) \rightarrow c_{5} \quad$ EXTEND and SIMPLIFY for 2
2(b). $h\left(c_{5}\right) \rightarrow c_{6}$
2(c). $\quad f\left(c_{2}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}\right) \rightarrow c_{7}$
2(d). $\quad c_{6} \approx c_{7}$
3(a). $f\left(c_{1}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{1}\right) \rightarrow c_{8}$
EXTEND and SIMPLIFY for 3
3(b). $\quad f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}\right) \rightarrow c_{9}$
3(c). $\quad h\left(c_{9}\right) \rightarrow c_{10}$
3(d). $\quad g\left(c_{3}, c_{10}\right) \rightarrow c_{11}$
$3(e) . \quad c_{8} \approx c_{11}$
4(a). $f\left(c_{1}, c_{2}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}, c_{2}\right) \rightarrow c_{12} \quad$ EXTEND and SIMPLIFY for 4
4(b). $\quad h\left(c_{12}\right) \rightarrow c_{13}$
4(c). $\quad f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{14}$
$4(d) . \quad c_{13} \approx c_{14}$
5(a). $f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{15} \quad$ EXTEND and SIMPLIFY for 5
5(b). $\quad f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{2}, c_{2}, c_{2}\right) \rightarrow c_{16}$
5(c). $\quad c_{15} \approx c_{16}$
6(a). $f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{2}, c_{2}, c_{2}\right) \rightarrow c_{17} \quad$ EXTEND and SIMPLIFY for 6
6(b). $\quad h\left(c_{17}\right) \rightarrow c_{18}$
6(c). $f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{1}, c_{2}\right) \rightarrow c_{19}$
$6(d) . \quad c_{18} \approx c_{19}$
7(a). $\quad f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}\right) \rightarrow c_{20}$
EXTEND and SIMPLIFY for 7

7(b). $\quad h\left(c_{20}\right) \rightarrow c_{21}$
7(c). $\quad f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{22}$
7(d). $\quad c_{21} \approx c_{22}$
8(a). $\quad c_{7} \approx c_{12}$ (Rule 2(c) is now removed.)
$8(b) . \quad c_{14} \approx c_{15}$ (Rule 4(c) is now removed.)
DEDUCE with 2(c) and 4(a)
$8(c) . \quad c_{16} \approx c_{17}$ (Rule 5(b) is now removed.)
DEDUCE with 4(c) and 5(a)
$8(d) . \quad c_{9} \approx c_{20}$ (Rule 3(b) is now removed.)
8(e). $\quad c_{19} \approx c_{20}$ (Rule 6(c) is now removed.)
DEDUCE with 5(b) and 6(a)
DEDUCE with 3(b) and 7(a)
$8(f) . \quad c_{4} \approx c_{22}$ (Rule $1(b)$ is now removed.)
DEDUCE with 6(c) and 7(a)
DEDUCE with 1(b) and 7(c)
We next orient equations into $C$-rules and apply other inference rules. The set of $C$-rules is $C=\left\{c_{3} \rightarrow\right.$ $c_{4}, c_{6} \rightarrow c_{7}, c_{8} \rightarrow c_{11}, c_{13} \rightarrow c_{14}, c_{15} \rightarrow c_{16}, c_{18} \rightarrow c_{19}, c_{21} \rightarrow c_{22}, c_{7} \rightarrow c_{12}, c_{14} \rightarrow c_{15}, c_{16} \rightarrow c_{17}, c_{9} \rightarrow$ $\left.c_{20}, c_{19} \rightarrow c_{20}, c_{4} \rightarrow c_{22}\right\}$. Using DEDUCE, COMPOSE, and ORIENT inference steps, it becomes $C^{\prime}=\left\{c_{3} \rightarrow c_{22}, c_{6} \rightarrow c_{12}, c_{8} \rightarrow c_{11}, c_{13} \rightarrow c_{17}, c_{15} \rightarrow c_{17}, c_{18} \rightarrow c_{20}, c_{21} \rightarrow c_{22}, c_{7} \rightarrow c_{12}, c_{14} \rightarrow c_{17}, c_{16} \rightarrow\right.$ $\left.c_{17}, c_{9} \rightarrow c_{20}, c_{19} \rightarrow c_{20}, c_{4} \rightarrow c_{22}\right\}$. The REWRITE inference step $9(\mathrm{~d})$ is available after the following inference steps $9(a), 9(b)$, and $9(c)$ :

| $9(a)$. | $h\left(c_{20}\right) \rightarrow c_{10}$ | COLLAPSE 3(c) with $c_{9} \rightarrow c_{20}$ |
| :--- | :--- | :--- |
| $9(b)$. | $c_{10} \rightarrow c_{22}$ | DEDUCE with 7(b) and 9(a), ORIENT, COMPOSE |
| $9(c)$. | $g\left(c_{22}, c_{22}\right) \rightarrow c_{11}$ | COLLAPSE 3(d) with $c_{3} \rightarrow c_{22}$ and $c_{10} \rightarrow c_{22}$ |
| $9(d)$. | $c_{11} \rightarrow c_{22}$ | REWRITE 9(c), ORIENT |

In the above, Rule 3(c) is removed after 9(a), Rule 9(a) is removed after 9(b), Rule 3(d) is removed after $9(c)$, and Rule $9(c)$ is removed after $9(d)$. We may obtain a congruence closure $R_{n}=C_{n} \cup D_{n}$ modulo $E \cup B$ for $P_{0}$ for some $n$ with some additional inference steps, where $C_{n}=\left\{c_{3} \rightarrow c_{22}, c_{4} \rightarrow\right.$ $c_{22}, c_{6} \rightarrow c_{12}, c_{7} \rightarrow c_{12}, c_{8} \rightarrow c_{11}, c_{9} \rightarrow c_{20}, c_{10} \rightarrow c_{22}, c_{11} \rightarrow c_{22}, c_{13} \rightarrow c_{17}, c_{14} \rightarrow c_{17}, c_{15} \rightarrow c_{17}, c_{16} \rightarrow$ $\left.c_{17}, c_{18} \rightarrow c_{20}, c_{21} \rightarrow c_{22}\right\}$ and $D_{n}$ consists of the following set of rules:
D1: $T \rightarrow c_{1}$
D2: $F \rightarrow c_{2}$
D3: $\perp \rightarrow c_{22}$
D4: $f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}\right) \rightarrow c_{5}$
D5: $h\left(c_{5}\right) \rightarrow c_{12}$
D6: $f\left(c_{1}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{1}\right) \rightarrow c_{22}$
D7: $f\left(c_{1}, c_{2}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}, c_{2}\right) \rightarrow c_{12}$
D8: $h\left(c_{12}\right) \rightarrow c_{17}$
D9: $f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{17}$
D10: $f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{2}, c_{2}, c_{2}\right) \rightarrow c_{17}$
D11: $h\left(c_{17}\right) \rightarrow c_{20}$
D12: $f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}\right) \rightarrow c_{20}$
D13: $h\left(c_{20}\right) \rightarrow c_{22}$
D14: $f\left(c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}, c_{1}\right) \rightarrow c_{22}$

Now we determine whether $h^{4}(f(F, F, F, F, F, F, F, F))$ is a fault state: i.e., $h^{4}(f(F, F, F, F, F, F, F, F))$ $\approx_{R_{n} \cup B \cup E}^{?} \perp$. Since $h^{4}(f(F, F, F, F, F, F, F, F)) \xrightarrow{*}_{R_{n}, E} h^{4}\left(f\left(c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}, c_{2}\right)\right) \rightarrow_{R_{n}, E} h^{4}\left(c_{5}\right) \rightarrow_{R_{n}, E}$ $h^{3}\left(c_{12}\right) \rightarrow_{R_{n}, E} h^{2}\left(c_{17}\right) \rightarrow_{R_{n}, E} h\left(c_{20}\right) \rightarrow_{R_{n}, E} c_{22}$ and $\perp \rightarrow_{R_{n}, E} c_{22}$, it is a fault state. Similarly, we can determine whether $f(T, F, F, F, F, F, F, T)$ is a fault state. Since $f(T, F, F, F, F, F, F, T) \xrightarrow{*} R_{R_{n}, E} f\left(c_{1}, c_{2}, c_{2}, c_{2}, c_{2}\right.$, $\left.c_{2}, c_{2}, c_{1}\right) \rightarrow_{R_{n}, E} c_{11} \rightarrow_{R_{n}, E} c_{22}$ and $\perp \rightarrow_{R_{n}, E} c_{22}$, it is a fault state. Meanwhile, $h^{2}(f(F, F, T, T, F, T, F, T))$ is not a fault state, i.e., $h^{2}(f(F, F, T, T, F, T, F, T)) \not \nsim_{R_{n} \cup B \cup E} \perp$. Since $h^{2}(f(F, F, T, T, F, T, F, T)) \xrightarrow{*} R_{R_{n} \cup B, E}$ $h^{2}\left(f\left(c_{2}, c_{2}, c_{1}, c_{1}, c_{2}, c_{1}, c_{2}, c_{1}\right)\right) \rightarrow_{R_{n} \cup B, E} h^{2}\left(c_{12}\right) \rightarrow_{R_{n} \cup B, E} h\left(c_{17}\right) \rightarrow_{R_{n} \cup B, E} c_{20}$ and $\perp \rightarrow_{R_{n} \cup B, E} c_{22}$, it is not a fault state.

## 5 Conclusion

We have presented a framework for constructing congruence closure modulo a finite set of permutation equations $E$, extending the abstract congruence closure framework for handling permutation function symbols with or without the interpreted function symbols (not necessarily permutation function symbols) satisfying each of the following properties: idempotency ( $I$ ), nilpotency ( $N$ ), unit ( $U$ ), $I \cup U$, or $N \cup U$. We have provided a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols by constructing congruence closure modulo $E$.

Although congruence closure procedures have been widely used in software/hardware verfication and satisfiability modulo theories (SMT) solvers, congruence closure procedures with built-in permutations have not been well studied. We believe that our framework for constructing congruence closure modulo permutation equations has practical significance to software/hardware verfication and SMT solvers involving built-in permutations, where built-in permutations are represented by a finite set of permutation equations containing permutation function symbols.

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[^0]:    ${ }^{1}$ If $g \in \mathscr{F}_{E}$, then $g$ is a commutative function symbol, i.e., $g\left(x_{1}, x_{2}\right) \approx g\left(x_{2}, x_{1}\right) \in E$.
    ${ }^{2}$ In this paper, $R \cup B, E$ (resp. $\left.R \cup B / E\right)$ denotes $(R \cup B), E$ (resp. $(R \cup B) / E$ ).

[^1]:    ${ }^{4}$ The only exception is the case where the REWRITE inference step using the nilpotency rule introduces constant 0 in the second component of the state, where 0 does not occur in $P_{0}$. But this requires at most one additional EXTEND inference step.
    ${ }^{5}$ The total number of symbols in $P_{i} \cup R_{i}$ for each transition step does not increase except by an EXTEND inference step, where an EXTEND inference step may increase this number by two.

