Congruence Closure Modulo Permutation Equations

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We present a framework for constructing congruence closure modulo permutation equations, which extends the *abstract congruence closure* [7] framework for handling permutation function symbols. Our framework also handles certain interpreted function symbols satisfying each of the following properties: idempotency (*I*), nilpotency (*N*), unit (*U*), $I \cup U$, or $N \cup U$. Moreover, it yields convergent rewrite systems corresponding to ground equations containing permutation function symbols. We show that congruence closure modulo a given finite set of permutation equations can be constructed in polynomial time using equational inference rules, allowing us to provide a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols.

1 Introduction

Congruence closure procedures [12, 18, 19] have been researched for several decades, and play important roles in software/hardware verification (see [9, 19, 20]) and satisfiability modulo theories (SMT) solvers [8, 10]. They provide fast decision procedures for determining whether a ground equation is an (equational) consequence of a given set of ground equations. (The fastest known congruence closure algorithm runs in $O(n \log n)$ [15].)

In [7, 14], some approaches to constructing the congruence closure of ground equations using completion methods were considered. These approaches capture the efficient techniques from standard term rewriting for congruence closure procedures. In particular, the *abstract congruence closure* approach in [7] (cf. Kapur's approach in [14]) constructs a reduced convergent ground rewrite system R_S for a finite set of ground equations S, which consists of either rewrite rules of the form $a \rightarrow c$ or $f(c_1, \ldots, c_n) \rightarrow c$ or $c \rightarrow d$ for fresh constants c_1, \ldots, c_n, c, d . Furthermore, R_S is a conservative extension of the equational theory induced by S (i.e. the congruence closure CC(S)) on ground terms, and two ground terms are congruent in CC(S) iff they have the same normal form w.r.t. R_S . Note that this approach does not require a total termination ordering on ground terms.

Congruence closure procedures were extended to congruence closure procedures modulo theories in order to handle interpreted function symbols in the signature [3,6,15]. The notion of congruence closure modulo associative and commutative (AC) theories was discussed in [6,16], and the notion of conditional congruence closure with uninterpreted and some interpreted function symbols was considered in [15].

Meanwhile, an equation is a *permutation equation* [1] if it is of the form $f(x_1,...,x_n) \approx f(x_{\pi(1)},...,x_{\pi(n)})$, where π is a permutation on the set $\{1,...,n\}$. Commutativity is the simplest case of permutation equations. Permutation equations are difficult to handle in equational reasoning without using the modulo approach. For example, an ordered completion procedure for *ordered rewriting* [5] produces every equation of the form $f(x_1, x_2, ..., x_n) \approx f(x_{\rho(1)}, x_{\rho(2)}, ..., x_{\rho(n)})$ (up to variable renaming) from two permutation equations $f(x_1, x_2, ..., x_n) \approx f(x_2, x_1, x_3, ..., x_n)$ and $f(x_1, x_2, ..., x_n) \approx f(x_2, x_3, ..., x_n, x_n)$, where ρ is a permutation in the symmetric group S_n of cardinality n!. (Recall that the symmetric group S_n can be generated by two cycles (12) and $(12 \cdots n)$.)

equations as "structural axioms" (defining a congruence relation on terms) rather than viewing them as "simplifiers" (defining a reduction relation on terms) [5].

In this paper, we present a framework for generating congruence closure modulo a finite set of permutation equations. To our knowledge, it has not been discussed in the literature, and a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols has not yet been known.

Our framework is based on the notion of abstract congruence closure that is particularly useful for term representation and checking *E*-equality between two flat terms for a given set of permutation equations *E*, which does not require an *E*-compatible ordering (cf. [17]). In addition, it also handles function symbols satisfying each of the following properties: idempotency (*I*), nilpotency (*N*), unit (*U*), $I \cup U$, or $N \cup U$. (If a function symbol is a permutation function symbol satisfying one of the above properties, then it should be a commutative function symbol.)

We show that congruence closure modulo a given finite set of permutation equations (with or without the function symbols satisfying the above properties) can be constructed in polynomial time, which provides a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols (appearing in E).

2 Preliminaries

We use the standard terminology and definitions of term rewriting [4, 11], congruence closure [7, 12, 19], and permutation groups [13]. We also use some terminology and the results of permutation equations found in [1, 2].

We denote by $\mathscr{T}(\mathscr{F}, \mathscr{X})$ the set of terms over a finite set of function symbols \mathscr{F} and a denumerable set of variables \mathscr{X} . We denote by $T(\mathscr{F})$ the set of ground terms over \mathscr{F} . We assume that each function symbol in \mathscr{F} has a fixed arity.

An *equation* is an expression $s \approx t$, where s and t are (first-order) terms built from \mathscr{F} and \mathscr{X} . A *ground equation* (resp. *ground term*) is an equation (resp. a term) which does not contain any variable.

We write s[u] if u is a subterm of s and denote by $s[t]_p$ the term that is obtained from s by replacing the subterm at position p of s by t.

An *equivalence* is a reflexive, transitive, and symmetric binary relation. An equivalence \sim on terms is a *congruence* if $s \sim t$ implies $u[s]_p \sim u[t]_p$ for all terms s, t, u and positions p.

An *equational theory* is a set of equations. We denote by \approx_E (called the *equational theory* induced by *E*) the least congruence on $T(\mathscr{F}, \mathscr{X})$ that is stable under substitutions and contains a set of equations *E*. If $s \approx_E t$ for two terms *s* and *t*, then *s* and *t* are *E*-equivalent.

Given a finite set $S = \{a_i \approx b_i \mid 1 \le i \le m\}$ of ground equations where $a_i, b_i \in T(\mathscr{F})$, the *congruence closure* CC(S) [3, 15] is the smallest subset of $T(\mathscr{F}) \times T(\mathscr{F})$ that contains S and is closed under the following rules: (i) $S \subseteq CC(S)$, (ii) for every $a \in T(\mathscr{F})$, $a \approx a \in CC(S)$ (*reflexivity*), (iii) if $a \approx b \in CC(S)$, then $b \approx a \in CC(S)$ (*symmetry*), (iv) if $a \approx b$ and $b \approx c \in CC(S)$, then $a \approx c \in CC(S)$ (*transitivity*), and (v) if $f \in \mathscr{F}$ is an *n*-ary function symbol (n > 0) and $a_1 \approx b_1, \ldots, a_n \approx b_n \in CC(S)$, then $f(a_1, \ldots, a_n) \approx f(b_1, \ldots, b_n) \in CC(S)$ (*monotonicity*). Note that CC(S) is also the equational theory induced by S.

A (strict) ordering \succ on terms is an irreflexive and transitive relation on $T(\mathscr{F}, \mathscr{X})$.

Given a rewrite system *R* and a set of equations *E*, the rewrite relation $\rightarrow_{R,E}$ on $T(\mathscr{F}, \mathscr{X})$ is defined by $s \rightarrow_{R,E} t$ if there is a non-variable position *p* in *s*, a rewrite rule $l \rightarrow r \in R$, and a substitution σ such that $s|_p \approx_E l\sigma$ and $t = s[r\sigma]_p$. The transitive and reflexive closure of $\rightarrow_{R,E}$ is denoted by $\stackrel{*}{\rightarrow}_{R,E}$. We say that a term *t* is an *R*,*E*-normal form if there is no term *t'* such that $t \rightarrow_{R,E} t'$. The rewrite relation $\rightarrow_{R/E}$ on $T(\mathscr{F}, \mathscr{X})$ is defined by $s \rightarrow_{R/E} t$ if there are terms u and v such that $s \approx_E u, u \rightarrow_R v$, and $v \approx_E t$. We simply say the rewrite relation $\rightarrow_{R/E}$ (resp. $\rightarrow_{R,E}$) on $T(\mathscr{F}, \mathscr{X})$ as the rewrite relation R/E (resp. R, E).

The rewrite relation R, E is *Church-Rosser modulo* E if for all terms s and t with $s \stackrel{*}{\leftrightarrow}_{R\cup E} t$, there are terms u and v such that $s \stackrel{*}{\rightarrow}_{R,E} u \stackrel{*}{\leftrightarrow}_{E} v \stackrel{*}{\leftarrow}_{R,E} t$. The rewrite relation R, E is *convergent modulo* E if R, E is Church-Rosser modulo E and R/E is well-founded.

The *depth* of a term *t* is defined as depth(t) = 0 if *t* is a variable or a constant and $depth(f(s_1, ..., s_n)) = 1 + \max\{depth(s_i) | 1 \le i \le n\}$. A term *t* is *flat* if its depth is 0 or 1.

An equation of the form $f(x_1,...,x_n) = f(x_{\rho(1)},...,x_{\rho(n)})$ is a *permutation equation* [1] if ρ is a permutation on $\{1,...,n\}$. We use variable naming in such a way that the left-hand side of each equation in a set of permutation equations with the same function symbol has the same name of variables $x_1,...,x_k$ from left to right. (In this paper, we assume that the set of function symbols \mathscr{F} in $T(\mathscr{F}, \mathscr{X})$ is finite and each function symbol in \mathscr{F} has a fixed arity.)

We denote by \mathscr{F}_E the set of all function symbols occurring in a finite set of permutation equations *E*.

If $e := f(x_1, ..., x_n) \approx f(x_{\rho(1)}, ..., x_{\rho(n)})$ is a permutation equation, then ρ is the permutation of this equation. We denote by $\pi[e]$ the permutation of e. For example, ρ is the permutation of the permutation equation $e' := f(x_1, x_2, x_3, x_4) \approx f(x_1, x_3, x_2, x_4)$ (i.e. $\pi[e'] = \rho$) with $\rho(1) = 1, \rho(2) = 3, \rho(3) = 2$, and $\rho(4) = 4$. Let *E* be a set of permutation equations with the same top function symbol. Then $\Pi[E]$ is defined as $\Pi[E] := {\pi[e] | e \in E}$. The permutation group generated by $\Pi[E]$ is denoted by $\langle \Pi[E] \rangle$.

Theorem 1. (see Theorem 1.4 in [2]) Let *E* be a set of permutation equations and let *e* be a permutation equation such that every equation in $E \cup \{e\}$ has the same (top) function symbol. Then $E \models e$ if and only if $\pi[e] \in \langle \Pi[E] \rangle$.

Let $i_1, i_2, ..., i_r$ ($r \le n$) be distinct elements of $I_n = \{1, 2, ..., n\}$. Then $(i_1 i_2 \cdots i_r)$, called *a cycle of length* r, is defined as the permutation that maps $i_1 \mapsto i_2$, $i_2 \mapsto i_3, ..., i_{r-1} \mapsto i_r$ and $i_r \mapsto i_1$, and every other element of I_n maps onto itself. The symmetric group S_n of cardinality n! can be generated by two cycles (12) and (12 $\cdots n$).

Example 1. Let $E = \{f(x_1, x_2, x_3, x_4, x_5) \approx f(x_2, x_1, x_3, x_4, x_5), f(x_1, x_2, x_3, x_4, x_5) \approx f(x_2, x_3, x_4, x_5, x_1)\}$. Then $\Pi[E]$ consists of two cycles $\{(12), (12345)\}$. Since two cycles (12) and (12345) generate the symmetric group S_5 , we see that $\langle \Pi[E] \rangle$ is S_5 . Therefore, $f(x_1, \dots, x_5) \approx_E f(x_{\tau(1)}, \dots, x_{\tau(5)})$ for any permutation $\tau \in S_5$ by Theorem 1.

Let *E* be a finite set of permutation equations. Then *E* can be uniquely decomposed as $\bigcup_{i=1}^{n} E_i$ such that (i) each E_i is a finite set of permutation equations, and (ii) E_j and E_k with $j \neq k$ are disjoint such that if $s_j \approx t_j \in E_j$ and $s_k \approx t_k \in E_k$, then s_j and s_k do not have the same top symbol (and are not variants of each other). Since we assume that each function symbol has a fixed arity, each distinct function symbol occurring in *E* corresponds to a distinct E_i in *E*. We denote by Eq(f) the corresponding equational theory with terms headed by such a function symbol *f*. Now, we may apply Theorem 1 for each Eq(f) in *E* with $f \in \mathscr{F}_E$.

3 Congruence closure modulo permutation equations

Definition 2. Let *K* be a set of constants disjoint from \mathscr{F} .

- (i) A *D*-rule (w.r.t. \mathscr{F} and *K*) is a rewrite rule of the form $f(c_1, \ldots, c_n) \to c$, where c_1, \ldots, c_n, c are constants in *K* and $f \in \mathscr{F}$ is an *n*-ary function symbol.
- (ii) A *C*-rule (w.r.t. *K*) is a rule $c \rightarrow d$, where *c* and *d* are constants in *K*.

In Definition 2(i), note that $f \in \mathscr{F}$ can also be a 0-ary function symbol (i.e. a constant).

Example 2. Let $E = \{f(x_1, x_2) \approx f(x_2, x_1), g(x_1, x_2, x_3) \approx g(x_2, x_1, x_3)\}$. If $\mathscr{F} = \{a, b, h, f, g\}$ with $\mathscr{F}_E = \{f, g\}$ and $P = \{f(b, g(b, a, a)) \approx h(a)\}$, then $D_0 = \{a \rightarrow c_0, b \rightarrow c_1, g(c_1, c_0, c_0) \rightarrow c_2, f(c_1, c_2) \rightarrow c_3, h(c_0) \rightarrow c_4\}$ is a possible set of *D*-rules over \mathscr{F} , and we have $K = \{c_0, c_1, c_2, c_3, c_4\}$. Using D_0 , we can simplify the original equations in *P*, which gives the set of *C* rules, i.e., $C_0 = \{c_3 \rightarrow c_4\}$ where $c_3 \succ c_4$.

Definition 3. Let *E* be a finite set of permutation equations and *K* be a set of constants disjoint from \mathscr{F} . A ground rewrite system $R = D \cup C$ is a *congruence closure modulo E* (w.r.t. \mathscr{F} and *K*) if the following conditions are met:

- (i) *D* is a set of *D*-rules and *C* is a set of *C*-rules, and for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\leftrightarrow}_{R,E} c$.
- (ii) R, E is a ground convergent (modulo E) rewrite system over $\mathscr{T}(\mathscr{F} \cup K)$.

In addition, given a set of ground equations *P* over $\mathscr{T}(\mathscr{F} \cup K)$, *R* is said to be a *congruence closure* modulo *E* (w.r.t. \mathscr{F} and *K*) for *P* if for all ground terms *s* and *t* over $\mathscr{T}(\mathscr{F})$, $s \stackrel{*}{\leftrightarrow}_{P \cup E} t$ iff there are ground terms *u* and *v* over $\mathscr{T}(\mathscr{F} \cup K)$ such that $s \stackrel{*}{\to}_{R,E} u \stackrel{*}{\leftrightarrow}_{R,E} t$.

In the following, by *B*-rules with the interpreted function symbol $g \in \mathscr{F}$, we mean either the idempotency rule (*I*): $\{g(x,x) \to x\}$ or the nilpotency rule (*N*): $\{g(x,x) \to 0\}$ or the unit rule (*U*): $\{g(x,0) \to x, g(0,x) \to x\}$ or $I \cup U$ or $N \cup U$.

Definition 4. Let *E* be a finite set of permutation equations and *K* be a set of constants disjoint from \mathscr{F} . A ground rewrite system $R = D \cup C$ is a *congruence closure modulo* $E \cup B$ (w.r.t. \mathscr{F} and *K*) if the following conditions are met:

- (i) *B* is a set of *B*-rules with the interpreted function symbol $g \in \mathscr{F}^{1}$.
- (ii) *D* is a set of *D*-rules and *C* is a set of *C*-rules, and for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\leftrightarrow}_{R,E} c$.
- (iii) $R \cup B, E$ is a convergent (modulo *E*) rewrite system over $\mathscr{T}(\mathscr{F} \cup K, \mathscr{X})$.²

In addition, given a set of ground equations *P* over $\mathscr{T}(\mathscr{F} \cup K)$, *R* is said to be a *congruence closure* modulo $E \cup B$ (w.r.t. \mathscr{F} and *K*) for *P* if for all ground terms *s* and *t* over $\mathscr{T}(\mathscr{F})$, $s \stackrel{*}{\leftrightarrow}_{P \cup B \cup E} t$ iff there are ground terms *u* and *v* over $\mathscr{T}(\mathscr{F} \cup K)$ such that $s \stackrel{*}{\rightarrow}_{R \cup B, E} u \stackrel{*}{\leftrightarrow}_{E} v \stackrel{*}{\leftarrow}_{R \cup B, E} t$.

Note that *B* or *E* can be empty in Definition 4. If *B* is empty, then it is the same as Definition 3. Also, condition (ii) in Definition 4 states that each constant *c* in *K* represents some term in $\mathscr{T}(\mathscr{F})$ w.r.t. *R*,*E*, meaning that *K* contains no superfluous constants (cf. [7]).

Definition 5. We denote by *W* the infinite set of constants $\{c_0, c_1, \ldots\}$ such that *W* is disjoint from \mathscr{F} , and denote by *K* a finite subset chosen from *W*. We define orderings \succ_K on *K*, and \succ and \succ_{lpo} on $\mathscr{T}(\mathscr{F} \cup K)$ as follows:

- (i) $c_i \succ_K c_j$ if i < j for all $c_i, c_j \in K$.
- (ii) $c \succ d$ if $c \succ_K d$, and $t \succ c$ if $t \rightarrow c$ is a *D*-rule.

¹If $g \in \mathscr{F}_E$, then g is a commutative function symbol, i.e., $g(x_1, x_2) \approx g(x_2, x_1) \in E$.

²In this paper, $R \cup B, E$ (resp. $R \cup B/E$) denotes $(R \cup B), E$ (resp. $(R \cup B)/E$).

- (iii) \succ_{lpo} is a lexicographic path ordering on $\mathscr{T}(\mathscr{F} \cup K)$, which can be defined from the following assumptions:
 - (iii.1) $c \succ_{lpo} d$ if $c \succ_K d$,
 - (iii.2) $t \succ_{lpo} c$ if t is any term headed by a function symbol f in \mathscr{F} and c is any constant in K, and (iii.3) there is a total precedence on symbols in \mathscr{F} .

Observe that \succ_{lpo} extends \succ , and is total on $\mathscr{T}(\mathscr{F} \cup K)$. (If the precedence on $\mathscr{F} \cup K$ is total, then the associated lexicographic path ordering \succ_{lpo} is total on $\mathscr{T}(\mathscr{F} \cup K)$ (see [11]).) We emphasize that a partial ordering \succ on $\mathscr{T}(\mathscr{F} \cup K)$ suffices for inference rules in Figure 1.

Figure 1 shows the inference rules for congruence closure modulo permutation equations, which extends the inference rules for the abstract congruence closure framework in [7]. We have the additional inference rule called the REWRITE rule in Figure 1. Also, we use the *E*-equality \approx_E instead of the equality \approx for the DEDUCE and DELETE inference rules. We write $(K, P, R) \vdash (K', P', R')$ to indicate that (K', P', R') can be obtained from (K, P, R) by application of an inference rule in Figure 1, where *K* denotes a set of new constants (see Definition 5), *P* a set of equations, and *R* a set of rewrite rules consisting of *C*-rules and *D*-rules. Also, in Figure 1, *B* denotes a set of *B*-rules. A *derivation* is a sequence of states $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$.

Lemma 6. If $(K, P, R) \vdash (K', P', R')$, then for all u and v in $\mathscr{T}(\mathscr{F} \cup K)$, we have $u \stackrel{*}{\leftrightarrow}_{E \cup B \cup P' \cup R'} v$ if and only if $u \stackrel{*}{\leftrightarrow}_{E \cup B \cup P \cup R} v$.

Proof. We consider each application of an inference rule τ for $(K, P, R) \vdash (K', P', R')$. If τ is EXTEND, SIMPLIFY, ORIENT, DELETE, COLLAPSE, or COMPOSE, then the conclusion can be verified similarly to [5,7].

If τ is REWRITE, then we let $P = \overline{P}$, $R = \overline{R} \cup \{l' \to r'\}$, $R' = \overline{R}$, $P' = \overline{P} \cup \{r\sigma \approx r'\}$, and K = K'. Since $(K \cup P \cup R) - (K' \cup P' \cup R') = \{l' \to r'\}$, we need to show that $l' \stackrel{*}{\leftrightarrow}_{E \cup B \cup P' \cup R'} r'$. As $l' = l\sigma \to_B r\sigma \leftrightarrow_{P'} r'$, we have $l' \stackrel{*}{\leftrightarrow}_{E \cup B \cup P' \cup R'} r'$. Conversely, since $(K' \cup P' \cup R') - (K \cup P \cup R) = \{r\sigma \approx r'\}$, we need to show that $r\sigma \stackrel{*}{\leftrightarrow}_{E \cup B \cup P \cup R} r'$. As $r\sigma \leftarrow_B l\sigma = l' \to_R r'$, we have $r\sigma \stackrel{*}{\leftrightarrow}_{E \cup B \cup P \cup R} r'$.

If τ is DEDUCE, then let $R = \overline{R} \cup \{s \to c, t \to d\}$, $P' = P \cup \{c \approx d\}$, $R' = \overline{R} \cup \{t \to d\}$, and K = K', where $s \approx_E t$. Since $(K \cup P \cup R) - (K' \cup P' \cup R') = \{s \to c\}$, we need to show that $s \stackrel{*}{\leftrightarrow}_{E \cup B \cup P' \cup R'} c$. As $s \stackrel{*}{\leftrightarrow}_E t \to_{R'} d \leftrightarrow_{P'} c$, we have $s \stackrel{*}{\leftrightarrow}_{E \cup B \cup P' \cup R'} c$. Conversely, since $(K' \cup P' \cup R') - (K \cup P \cup R) = \{c \approx d\}$, we need to show that $c \stackrel{*}{\leftrightarrow}_{E \cup B \cup P \cup R} d$. As $c \leftarrow_R s \stackrel{*}{\leftrightarrow}_E t \to_R d$, we have $c \stackrel{*}{\leftrightarrow}_{E \cup B \cup P \cup R} d$.

Definition 7. (i) A derivation is said to be *fair* if any inference rule that is continuously enabled is applied eventually.

(ii) By a *fair* μ -*derivation*, we mean that the EXTEND and SIMPLIFY rule are applied eagerly in a fair derivation.

Theorem 8. Let $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$ be a fair μ -derivation such that P_0 is a finite set of ground equations with $K_0 = \emptyset$ and $R_0 = \emptyset$.

(*i*) Each fair μ -derivation starting from the initial state (K_0, P_0, R_0) is finite.

(ii) If (K_n, P_n, R_n) is a final state (i.e. no inference rule can be applied to (K_n, P_n, R_n)) of a fair μ derivation starting from the initial state (K_0, P_0, R_0) , then $R_n \cup B, E$ is convergent modulo E, and R_n is a congruence closure modulo $E \cup B$ for P_0 .

Proof. Since $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$ is a fair μ -derivation, this derivation can be written as $(K_0, P_0, R_0) \vdash^* (K_m, P_m, R_m) \vdash (K_{m+1}, E_{m+1}, R_{m+1}) \vdash \cdots$, where the derivation $(K_m, P_m, R_m) \vdash (K_{m+1}, E_{m+1}, R_{m+1}) \vdash \cdots$ does not involve any application of the EXTEND rule, so we have the set $K_m = K_{m+1} = \cdots$.

EXTEND:
$$\frac{(K,P[t],R)}{(K \cup \{c\},P[c],R \cup \{t \to c\})}$$

if $t \to c$ is a D -rule, $c \in W - K$, and t occurs in some
equation in P .
SIMPLIFY:
$$\frac{(K,P[t],R \cup \{t \to c\})}{(K,P[c],R \cup \{t \to c\})}$$

if t occurs in some equation in P .³
REWRITE:
$$\frac{(K,P,R \cup \{t' \to t'\})}{(K,P \cup \{r\sigma \approx t'\},R)}$$

if $t' = l\sigma$, where $l \to r \in B$.
ORIENT:
$$\frac{(K,P,R \cup \{s \approx t\},R)}{(K,P,R \cup \{s \to t\})}$$

if $s \succ t$, and $s \to t$ is a D -rule or a C -rule.
DEDUCE:
$$\frac{(K,P,R \cup \{s \Rightarrow c, t \to d\})}{(K,P \cup \{c \approx d\},R \cup \{t \to d\})}$$

if $s \approx_E t$.
DELETE:
$$\frac{(K,P \cup \{s \approx t\},R)}{(K,P,R \cup \{t \to c, c \to d\})}$$

if $s \approx_E t$.
COMPOSE:
$$\frac{(K,P,R \cup \{t \to c, c \to d\})}{(K,P,R \cup \{t \to d, c \to d\})}$$

COLLAPSE:
$$\frac{(K,P,R \cup \{t[c] \to c', c \to d\})}{(K,P,R \cup \{t[d] \to c', c \to d\})}$$

if c is a proper subterm of t and $c \to d$ is a C -rule.

Figure 1: Inference rules for congruence closure modulo permutation equations

For (i), we provide a more concrete result in the following Lemma 9.

For (ii), let (K_n, P_n, R_n) be a final state of a fair μ -derivation starting from the state (K_0, P_0, R_0) . (Note that each fair μ -derivation starting from the initial state (K_0, P_0, R_0) is finite by (i), so we have some final state.) Observe that P_m, P_{m+1}, \ldots either contains only *C*-equations or is empty, and \succ can orient those *C* equations, so $P_n = \emptyset$. Since $l \succ_{lpo} r$ for all rules $l \rightarrow r \in R_n \cup B$, we see that $R_n \cup B/E$ is terminating.

Also, since $R_n \cup B$ is non-overlapping w.r.t. the rewrite system $R_n \cup B, E$ (i.e. there is no non-trivial critical pair between rules in $R_n \cup B$), $R_n \cup B, E$ is Church-Rosser modulo E by the critical pair lemma [5]. Thus, $R_n \cup B, E$ is convergent modulo E.

Finally, we show that for each constant $c \in K$, there exists at least one ground term $t \in \mathscr{T}(\mathscr{F})$ such that $t \stackrel{*}{\leftrightarrow}_{R_n,E} c$ by induction. Let c be a constant in K and $f(c_1,\ldots,c_k) \to c$ be the corresponding extension rule for c when c was added. By induction hypothesis, we have $s_i \stackrel{*}{\leftrightarrow}_{R_n,E} c_i$, and thus $f(s_1,\ldots,s_k) \stackrel{*}{\leftrightarrow}_{R_n,E} f(c_1,\ldots,c_k) \to_{\cup_i R_i} c$. By Lemma 6, we also have $f(s_1,\ldots,s_k) \stackrel{*}{\leftrightarrow}_{R_n\cup P_nB\cup E} c$, and thus $f(s_1,\ldots,s_k) \stackrel{*}{\leftrightarrow}_{R_n\cup B\cup E} c$ because $P_n = \emptyset$. As $R_n \cup B, E$ is convergent modulo E and no more REWRITE rule can be applied to $f(s_1,\ldots,s_k)$ by fairness of the derivation, we have $f(s_1,\ldots,s_k) \stackrel{*}{\leftrightarrow}_{R_n,E} c$. Thus R_n is a congruence closure modulo $E \cup B$ for P_0 .

In the following lemma, recall that function symbols in \mathscr{F} include 0-ary function symbols in \mathscr{F} , i.e., constants in \mathscr{F} .

Lemma 9. Let $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$ be a fair μ -derivation such that P_0 is a finite set of ground equations with $K_0 = \emptyset$ and $R_0 = \emptyset$. Then its derivation length is bounded by $O(n^2)$, where n is the sum of the sizes (number of symbols) of the left-hand and right-hand sides of equations in P_0 .

Proof. We show that the number of applications of each rule in Figure 1 in a fair μ -derivation is bounded above by $O(n^2)$, where *n* is the sum of the sizes (number of symbols in \mathscr{F}) of the left-hand and right-hand sides of equations in P_0 . Since $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$ is a fair μ -derivation, we may write this derivation as

$$(K_0, P_0, R_0) \vdash^* (K_m, P_m, R_m) \vdash (K_{m+1}, E_{m+1}, R_{m+1}) \vdash \cdots,$$

where the derivation $(K_m, P_m, R_m) \vdash (K_{m+1}, E_{m+1}, R_{m+1}) \vdash \cdots$ does not involve any application of the EXTEND rule, and thus we have the finite set $K_m = K_{m+1} = \cdots$.

- (i) The total number of the EXTEND inference steps is bounded by O(n). This is because the total number of \mathscr{F} -symbols in the second component of the state is not increasing for each transition step⁴ and each EXTEND inference step decreases this number by one.
- (ii) A derivation step by the SIMPLIFY, REWRITE, DEDUCE, COMPOSE, or COLLAPSE rule either reduces the number of function symbols of \mathscr{F} in $R_i \cup P_i$ or rewrites some constant. The length of a rewriting sequence $c_1 \rightarrow c_2 \rightarrow \cdots$ is bounded by $|K_m|$. (Here, $|K_m|$ is O(n) because the total number of the EXTEND inference steps is bounded by O(n) as discussed in (i).) Also, the total number of symbols in $P_i \cup R_i$ is bounded by $O(n + |K_m|)$,⁵ which is also O(n). This means that the total number of the SIMPLIFY, DEDUCE, COLLAPSE, or COMPOSE inference steps is bounded

⁴The only exception is the case where the REWRITE inference step using the nilpotency rule introduces constant 0 in the second component of the state, where 0 does not occur in P_0 . But this requires at most one additional EXTEND inference step.

⁵The total number of symbols in $P_i \cup R_i$ for each transition step does not increase except by an EXTEND inference step, where an EXTEND inference step may increase this number by two.

by $O(n^2)$. (Note that rewriting constants takes $O(n^2)$ because there are at most O(n) constants, and the length of a rewriting sequence for each constant is bounded by O(n).)

- (iii) The total number of the DELETE inference steps is bounded by $O(n + |K_m|)$ (i.e. O(n)) because the total number of symbols in $P_i \cup R_i$ is bounded by $O(n + |K_m|)$.
- (iv) The total number of the ORIENT inference steps is bounded by the total number of EXTEND, SIMPLIFY, DEDUCE, COLLAPSE, and COMPOSE inference steps, which is $O(n^2)$. Note that each ORIENT inference step neither increases the number of function symbols nor the number of constants.

Thus, the derivation length of any fair μ -derivation starting from (K_0, P_0, R_0) is bounded by $O(n^2)$.

Given a finite (fixed) set of permutation equations *E* and two terms $s = f(s_1, ..., s_k)$ and $t = f(t_1, ..., t_k)$ with $f \in \mathscr{F}_E$, we can determine whether $s \approx_E t$ in $O(n^2)$ time (measured in n = |s| + |t|) using an additional data structure (i.e. a table) that can be constructed in polynomial time [1]. If *s* and *t* are both flat, then we can determine whether $s \approx_E t$ in O(n) time using the following procedure with a table that can be constructed in polynomial time (see [1]).

Equality-Test(s, t)

Input: $s = f(c_1, ..., c_i)$ and $t = g(d_1, ..., d_j)$, where *s* and *t* are both flat. Output: If $s \approx_E t$, then return true. Otherwise, return false.

- 1. Determine whether s and t are headed by the same function symbol (i.e. f = g and thus i = j). If not, then return false. If it is true, then consider the following:
- 2. Determine whether $f \in \mathscr{F}_E$. If not, then *s* and *t* are compared by syntactic equality, and return true if they are syntactically equal. Otherwise, if $f \in \mathscr{F}_E$, then consider the following:
- 3. Determine whether $s \approx_E t$ using the *TestEq* procedure in [1].

It is easy to see that steps 1 and 2 of the *Equality-Test*(*s*, *t*) procedure take at most O(n) time. For step 3, which corresponds to the case f = g and $f \in \mathscr{F}_E$, it takes O(n) time for comparing two multisets.

If they are equal, then s and t are further compared in constant time using the *TestEq* procedure in [1] with a table that can be constructed in polynomial time. (Note that the arity of all $f \in \mathscr{F}_E$ and the size of the data structure (i.e. table) is bounded by a constant independent of the size of the input terms.) Therefore, the *Equality-Test*(s, t) procedure takes O(n) time using a table that can be constructed in polynomial time. In what follows, we denote by Table(Eq(f)) this table for each $f \in \mathscr{F}_E$ for a finite (fixed) set of permutation equations E.

Theorem 10. Given the table Table(Eq(f)) for each $f \in \mathscr{F}_E$, a congruence closure modulo $E \cup B$ for a finite set of ground equations P can be computed in $O(n^3)$ time, where n is the sum of the sizes (number of symbols) of the left and right sides of equations in P.

Proof. We first construct a fair μ -derivation $(K_0, P_0, R_0) \vdash (K_1, P_1, R_1) \vdash \cdots$ such that P_0 is a finite set of ground equations with $K_0 = \emptyset$ and $R_0 = \emptyset$.

It is easy to see that each EXTEND, SIMPLIFY, ORIENT, COMPOSE, and COLLAPSE inference step in the derivation takes O(n) time.

Each REWRITE inference step in the derivation takes O(n) time because we only need to consider for rules $g(x,x) \to x$, $g(x,x) \to 0$, $g(x,0) \to x$, and $g(0,x) \to x$ for some interpreted function symbol $g \in \mathscr{F}$.

Each DEDUCE inference step in the derivation takes O(n) time for checking *E*-equality (see the

Equality-Test(*s*, *t*) procedure) between two left-hand side terms *s* and *t* in R_i . Similarly, each DELETE inference step in the derivation takes O(n) time for checking *E*-equality.

By Lemma 9, we know that the derivation length of a fair μ -derivation is bounded by $O(n^2)$. Since each inference step in the derivation takes O(n) time, a congruence closure modulo $E \cup B$ for a finite set of ground equations P can be computed in $O(n^3)$ time.

Corollary 11. The word problem for a finite set of ground equations P with a fixed set of permutation function symbols is decidable in polynomial time.

Proof. We can decide whether $s \approx_E^? t$ for two ground terms *s* and *t* using a congruence closure modulo *E* for *P*. By Theorem 10, we can compute a congruence closure modulo *E* for *P* in polynomial time by constructing and using the table Table(Eq(f)) for each $f \in \mathscr{F}_E$. Let *R* be a congruence closure modulo *E* for *P*. We obtain each normal form of *s* and *t* using *R*. We first rewrite each constant symbol in \mathscr{F} of *s* and *t* to a new constant symbol in *K* obtained from constructing *R*, which takes O(m) time where m = |s| + |t|. Each rewrite step either reduces the size of a term or rewrites a constant in *K* to another constant in *K*. The length of a rewriting sequence $c_1 \rightarrow c_2 \rightarrow \cdots$ is bounded by |K| (i.e. O(n)), where *n* is the sum of the sizes of the left-hand and right-hand sides of equations in *P*. We may also infer that the sum of the sizes of the left-hand and right-hand sides of the rewrite rules in *R* is O(n+|K|), which is O(n). Each rewrite step takes at most $O(n^2)$ time using *R* and the *Equality-Test* procedure. By combining these steps together, we can decide whether $s \approx_E^? t$ for two ground terms *s* and *t* using their normal forms in polynomial time.

The above corollary also holds if some function symbols (not necessarily permutation function symbols) satisfies the properties, such as idempotency (*I*), nilpotency (*N*), unit (*U*), $I \cup U$, or $N \cup U$.

4 Example of congruence closure modulo $E \cup B$

Let *B* be the set of the equation for an idempotency function symbol *g*, i.e., $B = \{g(x, x) \rightarrow x\}$ and let *E* be the following set of permutation equations:

 $E = \{ f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx f(x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8), \\ f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx f(x_2, x_3, x_4, x_1, x_5, x_6, x_7, x_8), \\ f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx f(x_1, x_2, x_3, x_4, x_6, x_5, x_7, x_8), \\ f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx f(x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_7) \}.$

In this example, we may view each variable x_i as a switch in a specially designed electric board, where each variable will be assigned to either constant T (representing "on") or constant F (representing "off"). Each ground term $f(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$ with $c_i = T$ or F represents a certain state of this electric board. There is a special transformation button in this electric board, which may transform one state to another state of the electic board. This transformation button is represented by a function with symbol $h \notin \mathscr{F}_E$. The problem is to determine if a certain state in the electric board (represented by a term) generates a fault state (represented by term \perp). We see that $\prod[E] = \{(12), (1234), (56), (78)\}$, which means that $f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx_E f(x_{\rho(1)}, x_{\rho(2)}, x_{\rho(3)}, x_{\rho(4)}, x_5, x_6, x_7, x_8)$ for any permutation ρ on the set $\{1, 2, 3, 4\}$, $f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx_E f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ for any permutation π on the set $\{5, 6\}$, and $f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx_E f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ for any permutation τ on the set {7,8} (see Thereom 1). Therefore, eight switches in the board are partitioned into three components, i.e. { x_1, x_2, x_3, x_4 }, { x_5, x_6 } and { x_7, x_8 }, where the order of "switch on" or "switch off" does not matter in each component. For example, $f(T, T, F, F, T, F, T, F) \approx_E f(F, F, T, T, F, T, T, F)$. Meanwhile, g is an idempotent function symbol, which serves as a comparator for fault states. For example, if $g(\perp, f(F, F, F, T, T, T, T, F))$, then it is \perp if f(F, F, F, T, T, T, T, F) is \perp . Now we start with the following set of ground equations:

- 1. $f(T,T,T,T,T,T,T,T) \approx \bot$
- 2. $h(f(F,F,F,F,F,F,F,F,F)) \approx f(F,T,F,T,F,T,F,T)$
- 3. $f(T,F,F,F,F,F,F,T) \approx g(\perp,h(f(T,T,T,T,F,T,F,T)))$
- 4. $h(f(T,F,T,F,T,F,T,F)) \approx f(F,F,F,F,T,T,T,T)$
- 5. $f(F,F,F,F,T,T,T,T) \approx f(T,T,T,T,F,F,F,F)$
- 6. $h(f(T,T,T,T,F,F,F,F)) \approx f(T,T,T,T,T,F,T,F)$
- 7. $h(f(T,T,T,T,F,T,F,T)) \approx f(T,T,T,T,T,T,T,T)$

We show that, for example, each of $h^4(f(F,F,F,F,F,F,F,F,F))$ and f(T,F,F,F,F,F,F,F,F,T) is a fault state. (For notational brevity, by $h^i(t)$, we mean the function symbol h is applied to term $h^{i-1}(t)$ with $h^0(t)$ denoting t.) The initial state is (K_0, P_0, R_0) , where $K_0 = R_0 = \emptyset$ and P_0 consists of the above equations 1 - 7. We apply a fair μ -derivation starting with (K_0, P_0, R_0) and some intermediate and repetitive steps are omitted for clarity. In the following, each rewrite rule is an element of some R_i and each equation is an element of some P_j . We assume that $c_i \succ c_j$ if i < j.

1(a).	$T \to c_1, F \to c_2, \bot \to c_3$	EXTEND and SIMPLIFY for 1
1(b).	$f(c_1, c_1, c_1, c_1, c_1, c_1, c_1, c_1) \to c_4$	
1(c).	$c_4 \approx c_3$	
2(a).	$f(c_2, c_2, c_2, c_2, c_2, c_2, c_2, c_2) \to c_5$	EXTEND and SIMPLIFY for 2
2(b).	$h(c_5) \rightarrow c_6$	
. ,	$f(c_2, c_1, c_2, c_1, c_2, c_1, c_2, c_1) \to c_7$	
. ,	$c_6 \approx c_7$	
. ,	$f(c_1, c_2, c_2, c_2, c_2, c_2, c_2, c_1) \to c_8$	EXTEND and SIMPLIFY for 3
	$f(c_1, c_1, c_1, c_1, c_2, c_1, c_2, c_1) \to c_9$	
	$h(c_9) \rightarrow c_{10}$	
	$g(c_3,c_{10}) \to c_{11}$	
	$c_8 \approx c_{11}$	
	$f(c_1, c_2, c_1, c_2, c_1, c_2, c_1, c_2) \to c_{12}$	EXTEND and SIMPLIFY for 4
	$h(c_{12}) \rightarrow c_{13}$	
	$f(c_2, c_2, c_2, c_2, c_1, c_1, c_1, c_1) \to c_{14}$	
· · ·	$c_{13} \approx c_{14}$	
	$f(c_2, c_2, c_2, c_2, c_1, c_1, c_1, c_1) \to c_{15}$	EXTEND and SIMPLIFY for 5
	$f(c_1, c_1, c_1, c_1, c_2, c_2, c_2, c_2) \to c_{16}$	
	$c_{15} \approx c_{16}$	
. ,	$f(c_1, c_1, c_1, c_1, c_2, c_2, c_2, c_2) \to c_{17}$	EXTEND and SIMPLIFY for 6
. ,	$h(c_{17}) \rightarrow c_{18}$	
. ,	$f(c_1, c_1, c_1, c_1, c_1, c_2, c_1, c_2) \to c_{19}$	
	$c_{18} \approx c_{19}$	
(a).	$f(c_1, c_1, c_1, c_1, c_2, c_1, c_2, c_1) \to c_{20}$	EXTEND and SIMPLIFY for 7

7(b).	$h(c_{20}) \rightarrow c_{21}$	
7(c).	$f(c_1, c_1, c_1, c_1, c_1, c_1, c_1, c_1) \to c_{22}$	
7(d).	$c_{21} \approx c_{22}$	
8(<i>a</i>).	$c_7 \approx c_{12}$ (Rule 2(c) is now removed.)	DEDUCE with 2(c) and 4(a)
8(b).	$c_{14} \approx c_{15}$ (Rule 4(c) is now removed.)	DEDUCE with $4(c)$ and $5(a)$
8(c).	$c_{16} \approx c_{17}$ (Rule 5(b) is now removed.)	DEDUCE with 5(b) and 6(a)
8(d).	$c_9 \approx c_{20}$ (Rule 3(b) is now removed.)	DEDUCE with 3(b) and 7(a)
8(e).	$c_{19} \approx c_{20}$ (Rule 6(c) is now removed.)	DEDUCE with $6(c)$ and $7(a)$
8(f).	$c_4 \approx c_{22}$ (Rule 1(b) is now removed.)	DEDUCE with 1(b) and 7(c)

We next orient equations into *C*-rules and apply other inference rules. The set of *C*-rules is $C = \{c_3 \rightarrow c_4, c_6 \rightarrow c_7, c_8 \rightarrow c_{11}, c_{13} \rightarrow c_{14}, c_{15} \rightarrow c_{16}, c_{18} \rightarrow c_{19}, c_{21} \rightarrow c_{22}, c_7 \rightarrow c_{12}, c_{14} \rightarrow c_{15}, c_{16} \rightarrow c_{17}, c_9 \rightarrow c_{20}, c_{19} \rightarrow c_{20}, c_4 \rightarrow c_{22}\}$. Using DEDUCE, COMPOSE, and ORIENT inference steps, it becomes $C' = \{c_3 \rightarrow c_{22}, c_6 \rightarrow c_{12}, c_8 \rightarrow c_{11}, c_{13} \rightarrow c_{17}, c_{15} \rightarrow c_{17}, c_{18} \rightarrow c_{20}, c_{21} \rightarrow c_{22}, c_7 \rightarrow c_{12}, c_{14} \rightarrow c_{17}, c_{16} \rightarrow c_{17}, c_{19} \rightarrow c_{20}, c_{19} \rightarrow c_{20}, c_{4} \rightarrow c_{22}\}$. The REWRITE inference step 9(d) is available after the following inference steps 9(a), 9(b), and 9(c):

9(a).	$h(c_{20}) \to c_{10}$	COLLAPSE 3(c) with $c_9 \rightarrow c_{20}$
9(b).	$c_{10} \rightarrow c_{22}$	DEDUCE with 7(b) and 9(a), ORIENT, COMPOSE
9(c).	$g(c_{22}, c_{22}) \to c_{11}$	COLLAPSE 3(d) with $c_3 \rightarrow c_{22}$ and $c_{10} \rightarrow c_{22}$
9(d).	$c_{11} \rightarrow c_{22}$	REWRITE 9(c), ORIENT

In the above, Rule 3(*c*) is removed after 9(*a*), Rule 9(*a*) is removed after 9(*b*), Rule 3(*d*) is removed after 9(*c*), and Rule 9(*c*) is removed after 9(*d*). We may obtain a congruence closure $R_n = C_n \cup D_n$ modulo $E \cup B$ for P_0 for some *n* with some additional inference steps, where $C_n = \{c_3 \rightarrow c_{22}, c_4 \rightarrow c_{22}, c_6 \rightarrow c_{12}, c_7 \rightarrow c_{12}, c_8 \rightarrow c_{11}, c_9 \rightarrow c_{20}, c_{10} \rightarrow c_{22}, c_{11} \rightarrow c_{22}, c_{13} \rightarrow c_{17}, c_{14} \rightarrow c_{17}, c_{15} \rightarrow c_{17}, c_{16} \rightarrow c_{17}, c_{18} \rightarrow c_{20}, c_{21} \rightarrow c_{22}\}$ and D_n consists of the following set of rules:

D1: $T \rightarrow c_1$	D2: $F \rightarrow c_2$
D3: $\bot \rightarrow c_{22}$	D4: $f(c_2, c_2, c_2, c_2, c_2, c_2, c_2, c_2) \rightarrow c_5$
D5: $h(c_5) \rightarrow c_{12}$	D6: $f(c_1, c_2, c_2, c_2, c_2, c_2, c_2, c_1) \rightarrow c_{22}$
D7: $f(c_1, c_2, c_1, c_2, c_1, c_2, c_1, c_2) \rightarrow c_{12}$	D8: $h(c_{12}) \to c_{17}$
D9: $f(c_2, c_2, c_2, c_2, c_1, c_1, c_1, c_1) \rightarrow c_{17}$	D10: $f(c_1, c_1, c_1, c_1, c_2, c_2, c_2, c_2) \rightarrow c_{17}$
D11: $h(c_{17}) \to c_{20}$	D12: $f(c_1, c_1, c_1, c_1, c_2, c_1, c_2, c_1) \rightarrow c_{20}$
D13: $h(c_{20}) \rightarrow c_{22}$	D14: $f(c_1, c_1, c_1, c_1, c_1, c_1, c_1, c_1) \rightarrow c_{22}$

5 Conclusion

We have presented a framework for constructing congruence closure modulo a finite set of permutation equations E, extending the abstract congruence closure framework for handling permutation function symbols with or without the interpreted function symbols (not necessarily permutation function symbols) satisfying each of the following properties: idempotency (I), nilpotency (N), unit (U), $I \cup U$, or $N \cup U$. We have provided a polynomial time decision procedure for the word problem for a finite set of ground equations with a fixed set of permutation function symbols by constructing congruence closure modulo E.

Although congruence closure procedures have been widely used in software/hardware verfication and satisfiability modulo theories (SMT) solvers, congruence closure procedures with built-in permutations have not been well studied. We believe that our framework for constructing congruence closure modulo permutation equations has practical significance to software/hardware verfication and SMT solvers involving built-in permutations, where built-in permutations are represented by a finite set of permutation equations containing permutation function symbols.

References

- Jürgen Avenhaus (2004): Efficient Algorithms for Computing Modulo Permutation Theories. In David Basin & Michaël Rusinowitch, editors: Automated Reasoning - Second International Joint Conference, IJCAR 2004, Cork, Ireland, July 4–8, Springer, Berlin, Heidelberg, pp. 415–429, doi:10.1007/ 978-3-540-25984-8_31.
- [2] Jürgen Avenhaus & David A. Plaisted (2001): *General Algorithms for Permutations in Equational Inference*. Journal of Automated Reasoning 26(3), pp. 223–268, doi:10.1023/A:1006439522342.
- [3] Franz Baader & Deepak Kapur (2020): Deciding the Word Problem for Ground Identities with Commutative and Extensional Symbols. In Nicolas Peltier & Viorica Sofronie-Stokkermans, editors: Automated Reasoning, Springer International Publishing, Cham, pp. 163–180, doi:10.1007/978-3-030-51074-9_10.
- [4] Franz Baader & Tobias Nipkow (1998): *Term Rewriting and All That*. Cambridge University Press, Cambridge, UK, doi:10.1017/CB09781139172752.
- [5] Leo Bachmair (1991): Canonical Equational Proofs. Birkhäuser, Boston, doi:10.1007/ 978-1-4684-7118-2.
- [6] Leo Bachmair, IV Ramakrishnan, Ashish Tiwari & Laurent Vigneron (2000): Congruence Closure Modulo Associativity and Commutativity. In Hélène Kirchner & Christophe Ringeissen, editors: Frontiers of Combining Systems, Springer, Berlin, Heidelberg, pp. 245–259, doi:10.1007/10720084_16.
- [7] Leo Bachmair, Ashish Tiwari & Laurent Vigneron (2003): Abstract congruence closure. Journal of Automated Reasoning 31(2), pp. 129–168, doi:10.1023/B: JARS.0000009518.26415.49.
- [8] Clark Barrett & Cesare Tinelli (2018): Satisfiability Modulo Theories, pp. 305–343. Springer International Publishing, Cham, doi:10.1007/978-3-319-10575-8_11.
- [9] David Cyrluk, Sreeranga Rajan, Natarajan Shankar & Mandayam K. Srivas (1995): Effective theorem proving for hardware verification. In Ramayya Kumar & Thomas Kropf, editors: Theorem Provers in Circuit Design, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 203–222, doi:10.1007/3-540-59047-1_50.
- [10] Leonardo De Moura & Nikolaj Bjørner (2011): Satisfiability modulo theories: introduction and applications. Communications of the ACM 54(9), pp. 69–77, doi:10.1145/1995376.1995394.
- [11] Nachum Dershowitz & David A. Plaisted (2001): *Rewriting*. In: *Handbook of Automated Reasoning*, chapter 9, Volume I, Elsevier, Amsterdam, pp. 535 610, doi:10.1016/b978-044450813-3/50011-4.

- [12] Peter J. Downey, Ravi Sethi & Robert Endre Tarjan (1980): Variations on the Common Subexpression Problem. J. ACM 27(4), p. 758–771, doi:10.1145/322217.322228.
- [13] Thomas W. Hungerford (1980): Algebra. Springer, New York, NY, doi:10.1007/978-1-4612-6101-8.
- [14] Deepak Kapur (1997): Shostak's Congruence Closure as Completion. In Hubert Comon, editor: Rewriting Techniques and Applications, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 23–37, doi:10.1007/ 3-540-62950-5_59.
- [15] Deepak Kapur (2019): Conditional Congruence Closure over Uninterpreted and Interpreted Symbols. Journal of Systems Science and Complexity 32, pp. 317–355, doi:10.1007/s11424-019-8377-8.
- [16] Deepak Kapur (2021): A Modular Associative Commutative (AC) Congruence Closure Algorithm. In: 6th International Conference on Formal Structures for Computation and Deduction, FSCD 2021, Buenos Aires, Argentina (Virtual Conference), July 17–24, 195, LIPIcs, pp. 15:1–15:21, doi:10.4230/LIPIcs.FSCD.2021. 15.
- [17] Dohan Kim & Christopher Lynch (2021): An RPO-based ordering modulo permutation equations and its applications to rewrite systems. In: 6th International Conference on Formal Structures for Computation and Deduction, FSCD 2021, Buenos Aires, Argentina (Virtual Conference), July 17–24, 195, LIPIcs, pp. 19:1–19:17, doi:10.4230/LIPIcs.FSCD.2021.19.
- [18] Dexter Kozen (1977): Complexity of Finitely Presented Algebras. In John E. Hopcroft, Emily P. Friedman & Michael A. Harrison, editors: Proceedings of the 9th Annual ACM Symposium on Theory of Computing, May 4-6, 1977, Boulder, Colorado, USA, ACM, pp. 164–177, doi:10.1145/800105.803406.
- [19] Greg Nelson & Derek C. Oppen (1980): Fast Decision Procedures Based on Congruence Closure. J. ACM 27(2), p. 356–364, doi:10.1145/322186.322198.
- [20] Vilhelm Sjöberg & Stephanie Weirich (2015): *Programming up to Congruence*. *SIGPLAN Not*. 50(1), p. 369–382, doi:10.1145/2676726.2676974.