Infinite games with uncertain moves

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We study infinite two-player games where one of the players is unsure about the set of moves available to the other player. In particular, the set of moves of the other player is a strict superset of what she assumes it to be. We explore what happens to sets in various levels of the Borel hierarchy under such a situation. We show that the sets at every alternate level of the hierarchy jump to the next higher level.

1 Introduction

Infinte two-player games have attracted a lot of attention and found numerous applications in the fields of topology, descriptive set-theory, computer science etc. Examples of such types of games are: Banach-Mazur games, Gale-Stewart games, Wadge games, Lipschitz games, etc. [7, 6, 11, 3], and they each characterize different concepts in descriptive set theory.

These games are typically played between two players, Player 0 and Player 1, who take turns in choosing finite sequences of elements (possibly singletons) from a fixed set A (finite or infinite) which is called the alphabet. This process goes on infinitely and hence defines an infinite sequence $u_0u_1u_2\ldots$ of finite strings which in itself is an infinite string over the set A. In addition, the game has a winning condition *Win* which is a subset of the set of infinite strings over A, A^ω. Player 0 is said to win the game if the sequence $u_0u_1u_2\ldots$ is in *Win*. Player 1 wins otherwise.

In addition to their applications in descriptive set-theory and topology, such games have also been used in computer science in the fields of verification and synthesis of reactive systems [4]. The verification problem is modeled as a game between two players: the system player and the environment player. The winning set *Win* is specified using formulas in some logic, LTL, CTL, μ-calculus etc. The goal of the system player is to meet the specification along every play and that of the environment player is to exhibit a play which does not meet it. To verify the system then amounts to show that the system player has a winning strategy in the underlying game and to find this strategy.

When *Win* is specified using the usual logics, it corresponds to sets in the low levels of the Borel hierarchy. It is known that the complexity of the winning strategy increases with the increase in the level of the Borel hierarchy to which *Win* belongs [10]. For instance, in Gale-Stewart games, reachability, safety and Muller are winning conditions in the Σ^0_1, Π^0_1 and Σ^0_2 levels of the Borel hierarchy respectively and a player has positional winning strategies for reachability and safety but needs memory to win for the Muller condition. However it was shown in [5, 8] that a finite amount of memory suffices. The notion of Wadge reductions also formalises this increase in complexity of the sets along the Borel hierarchy.

Such games (esp. Banach-Mazur and Gale-Stewart games) also find applications in linguistics. [2] shows that conversations have a topological structure similar to that of Banach-Mazur games and explores how the different types of objectives of conversations correspond to different levels in the Borel hierarchy

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depending on their complexity. [2] also applied of the classical results from the literature of Banach-Mazur games to the conversational setting. [1] applies Gale-Stewart games to the study of politeness.

In this paper, we look at what happens to sets in the Borel hierarchy when the underlying alphabet is expanded. That is, the alphabet is changed from A to B such that B is a strict superset of A. We show that sets at every alternate level of the Borel hierarchy undergo a jump to the next higher level. More precisely, a set at level n of the hierarchy with alphabet A moves to level $n+1$ when the alphabet is expanded to B. This process goes on for all countable levels and stabilises at ω.

Our result has consequences for both formal verification and linguistic applications some of which we elucidate in the concluding section.

The rest of the paper is organised as follows. In Section 2 we formally introduce the necessary concepts and give the required background for the paper. Then in Section 3 we state and prove the main results of the paper. Finally we conclude with some interesting consequences in Section 4.

2 Preliminaries

In this section we present the necessary background required for the paper. Although we define most of the concepts used in the paper, we assume some familiarity with the basic notions of topology and set-theory.

2.1 Open and closed sets

Let A be a non-empty set. We sometimes refer to A as the alphabet. For any subset X of A, as usual, we denote by X^* the set of finite strings over X and by X^ω, the set of countably infinite strings over X. For any string $u \in A^* \cup A^\omega$ we denote the ith element of u by $u(i)$. The set of prefixes of u are all strings $v \in A^*$ such that $u = vv'$ for $v' \in A^* \cup A^\omega$.

We define a topology on A^ω, the standard topology (also known as the Cantor topology) on the set of infinite strings over A. This topology can be defined in at least three equivalent ways. The first way is to define the discrete topology on A and then assign A^ω the product topology. The second way is to explicitly define the open sets of the topology. The open sets are given by sets of the form XA^ω where X is a subset of A^*. Thus an open set is a set of finite strings over X followed by their all possible continuations. For a set $X \subseteq A^*$, we denote the open set XA^ω by $O_A(X)$ or simply by $O(X)$ when the underlying alphabet A is clear from the context. When X is a singleton $\{u\}$, we abuse notation to denote the open set uA^ω by $O_A(u)$. Example 1 illustrates these concepts.

Example 1. Let $A = \{a,b,c\}$. Then $abcA^\omega$ is an open set and so is $abA^\omega \cup baA^\omega$. The complement of the set $abcA^\omega$ is the set X of all strings that do not have abc as their prefix. This is a closed set.

Yet another equivalent way to define the topology is to give an explicit metric for it. Given two strings, $u_1, u_2 \in A^\omega$, the distance between them $d(u_1,u_2)$ is defined to be $1/2^{n(u_1,u_2)}$, where $n(u_1,u_2)$ is the first index where u_1 and u_2 differ from each other. Thus the above topology is metrisable. Henceforth, when we use the term ‘set’ we shall mean a subset of A^ω.

Note that the set $(\overline{abcA^\omega})$ in the above example is also open. That is because it is a union of the open sets $O(aa), O(ac), O(b)$ and $O(c)$. Such sets, which are both open and closed are called clopen sets. So what is a set which is open but not closed (and vice versa)?

Proposition 1 ([9]) *If A is a finite alphabet, a subset of A^ω is clopen if and only if it is of the form XA^ω where X is a finite subset of A^*.*

Thus if A is finite then a set of the form XA^ω where X is an infinite subset of A^* is open but not closed. If A is infinite, the subsets of A^ω of the form XA^ω, where X is a set of words of bounded length of A^* are clopen. However there might exist clopen sets which are not of this form.

2.2 The Borel hierarchy

A set of subsets of A^ω is called a σ-algebra if it is closed under countable unions and complements. Given a set X, the smallest σ-algebra containing X is called the σ-algebra generated by X. It is equivalent to the intersection of all the σ-algebras containing X. The sigma algebra generated by the open sets of a topological space is called the Borel σ-algebra and its sets are called the Borel sets.

The Borel sets can also be defined inductively. This gives a natural hierarchy of classes Σ^0_α and Π^0_α for $1 \leq \alpha < \omega_1$. Let Σ^0_1 be the set of all open sets. $\Pi_1 = \overline{\Sigma^0_1}$ is the set of all closed sets. Then for any $\alpha > 1$ where α is a successor ordinal, define Σ^0_α to be the countable union of all $\Pi^0_{\alpha-1}$ sets and define Π^0_α to be the complement of Σ^0_α. For a limit ordinal η, $1 < \eta < \omega_1$, Σ^0_η is defined as $\Sigma^0_\eta = \bigcup_{\alpha<\eta} \Sigma^0_\alpha$ and $\Pi^0_\eta = \overline{\Sigma^0_\eta}$. The infinite hierarchy thus generated is called the Borel hierarchy and they together form the Borel algebra. It is known [9] that if the space is metrisable and the underlying alphabet contains at least two elements, then the hierarchy is indeed infinite, that is, the containments, $\Sigma^0_\alpha \subset \Sigma^0_{\alpha+1}$ and $\Pi^0_\alpha \subset \Pi^0_{\alpha+1}$ are strict.

2.3 Wadge reductions and complete sets

Let A and B be two alphabets. A function $f : A^\omega \to B^\omega$ is said to be continuous if for every open subset $Y \subseteq B^\omega$, $f^{-1}(Y)$ is also open.

A set $X \subseteq A^\omega$ is said to Wadge reduce to another set $Y \subseteq B^\omega$, denoted $X \leq_W Y$, if there exists a continuous function $f : A^\omega \to B^\omega$ such that $f^{-1}(Y) = X$.

Let A be an alphabet. A set $X \subseteq A^\omega$ is said to be Σ^0_α (resp. Π^0_α) complete if $X \in \Sigma^0_\alpha$ (resp. $X \in \Pi^0_\alpha$) and for any other alphabet B and for any Σ^0_α (resp. Π^0_α) set $Y \subseteq B^\omega$, $Y \leq_W X$. Intuitively, given a class of sets Γ, the complete sets of that class represent the sets which are structurally the most complex in that class.

For the Borel hierarchy, completeness can be characterised in the following simple way:

Proposition 2 ([9]) *Let $X \subseteq A^\omega$. Then X is Π^0_α (resp. Σ^0_α) complete if and only if $X \in \Pi^0_\alpha \setminus \Sigma^0_\alpha$ (resp. $\Sigma^0_\alpha \setminus \Pi^0_{\alpha-1}$).*

2.4 Infinite games

Let A be an alphabet. An infinite game on A is played between two players, Player 0 and Player 1, who take turns in choosing finite sequences of elements (possibly singletons) from a fixed set A (finite or infinite) which is called the alphabet. This process goes on infinitely and hence defines an infinite sequence $u_0u_1u_2\ldots$ of finite strings which in itself is an infinite string over the set A. In addition, the game has a winning condition *Win* which is a subset of the set of infinite strings over A, A^ω. Player 0 is said to win the game if the sequence $u_0u_1u_2\ldots$ is in *Win*. Player 1 wins otherwise.

In a Banach-Mazur game, each player at her turn chooses a finite non-empty sequence of elements from A while in a Gale-Stewart game the players are restricted to choosing just single elements from A. An infinite game can also be imagined to be played on a graph $G = (V, E)$ where the set of vertices V is partitioned into V_0 and V_1 which represent the Player 0 and 1 vertices respectively. The game starts at an initial vertex $v_0 \in V$ and the players take turns in moving a token along the edges of the graph depending on whose vertex it is currently. This process is continued ad infinitum and thus generates an infinite path p in the graph G. Player 0 wins if and only if $p \in Win$ where Win is a pre-specified set of infinite paths.

3 Results

In this section we present the main results of this paper. Given a subset B of an alphabet A the topology of B^ω where the open sets are given by $O \cap B^\omega$ for every open set O of A^ω is called the relative topology of B^ω with respect to A^ω. However we are interested in the opposite question. What happens when the alphabet expands? In particular, we show that when the alphabet set changes from A to B (say) such that B is a strict superset of A then the sets in the alternative levels of the Borel hierarchy undergo a jump in levels.

Lemma 1 *Let A and B be two alphabets such that $A \subsetneq B$. An open set O in the space A^ω jumps to Σ_2^0 in the space B^ω. A closed set C in the space A^ω remains closed in B^ω.*

Proof The proof is by carried out by coding the open set O in the space B^ω and demonstrating a complete set for B^ω.

Let O be an open set in A^ω. Then O is of the form XA^ω where $X \subseteq A^*$. Let $\mathscr{X}_\beta$ be an indexing of the set X.

Each element u of X gives the open set $O_A(u)$ which is a subset of A^ω. Now, when we move to the alphabet B, the set $O_B(u)$ is the set of strings which have u as a prefix and all possible continuations using letters of B. Thus $O_B(u)$ is a strict superset of $O_A(u)$. Hence, we need to restrict $O_B(u)$ in B^ω such that we obtain a set which is equal to $O_A(u)$ in A^ω. One way to do do so is as follows. Consider all the finite continuations of u in letters from A. Let $\mathscr{U}_\gamma$ be an indexed set of all these continuations. Then $O_A(u)$ is the set

$$O_A(u) = \bigcap O_B(u'),\ u' \in \mathscr{U}_\gamma \tag{1}$$

which is a closed set, being an arbitrary intersection of closed sets.

Thus the set O can be represented in B^ω as

$$O = \bigcup O_A(u),\ u \in \mathscr{X}_\beta$$

each of which by (1) is a closed set. Hence $O \in \Sigma_2^0$ in the space B^ω.

Next we demonstrate a Σ_1^0 set O in a space A^ω which is complete for Σ_2^0 in a space B^ω where $A \subsetneq B$. Let $A = \{a,b\}$ and $B = \{a,b,c\}$. Let $X = \{ab, abab, ababab, \ldots\} \subset A^*$ and let $O = XA^\omega$. Then O is open. Each subset $O_A(u),\ u \in X$ is represented in B^ω as

$$O_A(u) = O_B(u) \cap O_B(ua) \cap O_B(ub) \cap O_B(uaa) \cap O_B(uab) \cap O_B(uba) \cap O_B(ubb) \cap \ldots$$

and

$$O = O_A(u_1) \cup O_A(u_2) \cup \ldots,\ u_i \in X$$

Hence O is a Σ_2^0 set in B^ω.

To show that O is Σ_2^0 complete for B^ω we use Proposition 2. O is not open in B^ω. Indeed, because otherwise, there exists a finite string u whose all possible continuations with letters from B are in O and that is a contradiction. O is also not closed in B^ω. To see this, note that the complement of O, $\overline{O}$ in A^ω is the set XA^ω where $X \subseteq A^*$ is given as $X = \{b, aa, abb, abaa, \ldots\}$. For O to be closed in B^ω, $\overline{O}$ should be open in B^ω. This means that there should exist a finite string v whose all possible continuations with letters from B are in $\overline{O}$ which is again a contradiction.

Thus $O \notin \Sigma_1^0$ and $O \notin \Pi_1^0$ in B^ω and hence it is complete for Σ_2^0 in B^ω.

Next suppose C is a closed set in A^ω. We show how to represent C in B^ω. Let $\mathscr{U}_\beta$ be the indexed set of prefixes of C. Then C can be represented in B^ω as

$$C = \bigcap O_B(v),\ v \in \mathscr{U}_\beta$$

Each $O_B(v)$ is a closed set in B^ω and hence C being an arbitrary intersection of closed sets in B^ω is closed. Thus $C \in \Pi_1^0$ in A^ω remains Π_1^0 in B^ω. ∎

We generalise the above Lemma to the entire Borel hierarchy in the following theorem.

Theorem 1 *Let A and B be two alphabets such that $A \subsetneq B$. We have the following in the Borel hierarchy:*

1. *For $1 \leq \alpha < \omega$ and α odd,*
   (a) *a set $X \in \Sigma_\alpha^0$ in the space A^ω jumps to $\Sigma_{\alpha+1}^0$ in the space B^ω*
   (b) *a set $X \in \Pi_\alpha^0$ in the space A^ω remains Π_α^0 in the space B^ω.*
2. *For $1 \leq \alpha < \omega$ and α even,*
   (a) *a set $X \in \Sigma_\alpha^0$ in the space A^ω remains Σ_α^0 in the space B^ω*
   (b) *a set $X \in \Pi_\alpha^0$ in the space A^ω jumps to $\Pi_{\alpha+1}^0$ in the space B^ω.*
3. *For $\alpha \geq \omega$, a Σ_α^0 (resp. Π_α^0) set remains Σ_α^0 (resp. Π_α^0) on going from the space A^ω to B^ω. That is, the sets stabilise.*

Proof The proof is by induction on α. For the base case, $\alpha = 1$, the result follows from Lemma 1.

The inductive case is relatively straightforward, given the inductive structure of the Borel hierarchy. For convenience, we subscript the sets with A or B to denote whether they are sets in A^ω or B^ω respectively.

Suppose $1 < \alpha < \omega$ and α is odd. Then

$$\begin{aligned}\Sigma^0_{\alpha,X} =& \bigcup \Pi^0_{\alpha-1,X} \text{ [by definition]}\\ =& \bigcup \Pi^0_{\alpha,Y} \text{ [by induction hypothesis]}\\ =& \Sigma^0_{\alpha+1,Y}\end{aligned}$$

$$\begin{aligned}\Pi^0_{\alpha,X} =& \overline{\Sigma}^0_{\alpha,X} = \overline{\bigcup \Pi^0_{\alpha-1,X}} = \bigcap \overline{\Pi}^0_{\alpha-1,X} = \bigcap \Sigma^0_{\alpha-1,X} \text{ [by definition]}\\ =& \bigcap \Sigma^0_{\alpha-1,Y} \text{ [by induction hypothetis]}\\ =& \Pi^0_{\alpha,Y}\end{aligned}$$

Now, suppose $1 < \alpha < \omega$ and α is even. Then

$$\begin{aligned}\Sigma^0_{\alpha,X} =& \bigcup \Pi^0_{\alpha-1,X} \text{ [by definition]}\\ =& \bigcup \Pi^0_{\alpha-1,Y} \text{ [by induction hypothesis]}\\ =& \Sigma^0_{\alpha,Y}\end{aligned}$$

$$\begin{aligned}\Pi^0_{\alpha,X} =& \overline{\Sigma}^0_{\alpha,X} = \overline{\bigcup \Pi^0_{\alpha-1,X}} = \bigcap \overline{\Pi}^0_{\alpha-1,X} = \bigcap \Sigma^0_{\alpha-1,X} \text{ [by definition]}\\ =& \bigcap \Sigma^0_{\alpha,Y} \text{ [by induction hypothetis]}\\ =& \Pi^0_{\alpha+1,Y}\end{aligned}$$

Finally,

$$\Sigma^0_{\omega,X} = \bigcup_{n<\omega} \Sigma^0_{n,X} = \bigcup_{n<\omega} \Sigma^0_{n,Y} = \Sigma^0_{\omega,Y}$$

and

$$\Pi^0_{\omega,Y} = \overline{\Sigma}^0_{\omega,Y} = \Pi^0_{\omega,X}$$

∎

The above result can be concisely summarised by Figure 1.

$$\begin{array}{l} \Sigma^0_1 \longrightarrow \overset{\circlearrowright}{\Sigma^0_2} \qquad \Sigma^0_3 \longrightarrow \overset{\circlearrowright}{\Sigma^0_4} \cdots\cdots \overset{\circlearrowright}{\Sigma^0_\omega} \qquad \overset{\circlearrowright}{\Sigma^0_{\omega+1}} \cdots\cdots \overset{\circlearrowright}{\Sigma^0_{\omega_1}} \\ \\ \overset{\circlearrowright}{\Pi^0_1} \qquad \Pi^0_2 \longrightarrow \overset{\circlearrowright}{\Pi^0_3} \qquad \Pi^0_4 \cdots\cdots \overset{\circlearrowright}{\Pi^0_\omega} \qquad \overset{\circlearrowright}{\Pi^0_{\omega+1}} \cdots\cdots \overset{\circlearrowright}{\Pi^0_{\omega_1}} \end{array}$$

Figure 1: Jumps in the Borel hierarchy

4 Applications

The result we showed has interesting consequences in the fields of both formal verification and linguistics.

4.1 Formal verification

As we mentioned in the introduction, to formally verify a reactive system M (a piece of hardware or software which interacts with users/environment), we often model the system as a finite graph $G(M)$. Two players, the system player and the environment player then play an infinite game on $G(M)$. The goal of the system player is to meet a certain specification on all plays on $G(M)$ and that of the environment player is to exibit a play which does not meet it.

The result stated in this paper represents situations where the system player is unsure about the exact moves of the environment player. This shows that in such a situation, the system player might have to strategise at a higher level of the hierarchy in order to account for this uncertainty.

It can also be used to represent situations where the underlying model might change (expand). Let M be the original system and M' be the expanded system (which is generated from M by the addition of a module say). If the objective of the system player in $G(M)$ was to reach one of the states in some subset R of $G(M)$ (reachability) then it is enough for her to play positionally. However, in the bigger graph $G(M')$ she not only has to reach R but also has to stay within the states of the original graph $G(M)$ in order to achieve the same objective. This is the Muller objective which is a level higher.

Example 2. Consider the example shown in Figure 2. Player 0 nodes have been depicted as $\bigcirc$ and Player 1 nodes as $\Box$. Suppose initially the system is M and the objective of Player 1 in $G(M)$ is to reach v_3. Then the winning set is the set of all sequences in $V = \{v_0, v_1, v_2, v_3\}$ in which v_3 occurs in some position. That is, $Win = \{u \mid \exists i,\ u(i) = v_3\}$. This is a reachability condition where the reachability set $R = \{v_3\}$. To win, Player 0 can either play v_1 or v_2 from v_0 and hence both these strategies are winning strategies for her. Now suppose the system expands to M' where, in $G(M')$, it is possible for Player 1 to go to the new node v_4 from v_1. Also suppose Win remains the same. Then Win is no longer a reachability condition because then it would also include sequences involving the vertex v_4. It is rather a Muller condition where the Muller set $\mathscr{F} = \{\{v_0, v_1, v_2, v_3\}\}$. However, note that Player 0 does not have a winning strategy in this game. That is because to win, she has to visit vertex v_1 infinitely often from which Player 1 can force the play through v_4 infinitely often.

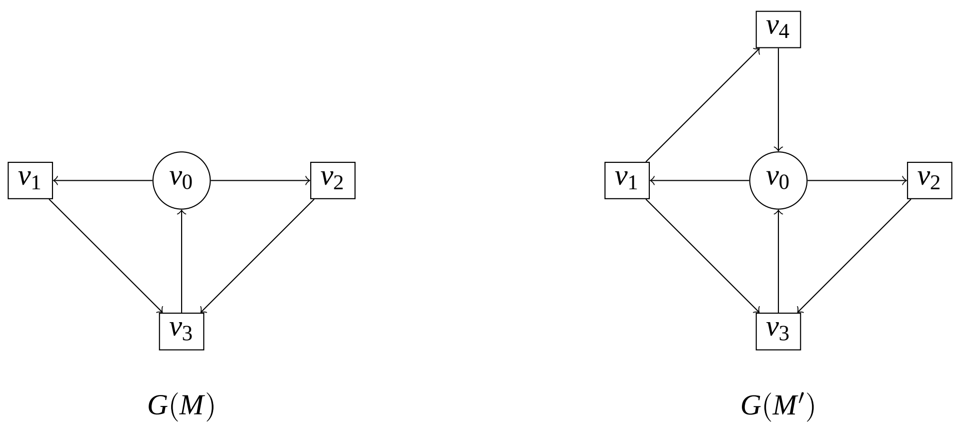

Figure 2: Jump from reachability to Muller

4.2 Linguistics

In [2] we demonstrated what seems to be a compelling similarity between human conversations and Banach-Mazur games. We showed how various conversational objectives correspond to various levels of the Borel hierarchy and how strategies of increasing complexity are called for to attain such objectives. Our result shows that when Player 1 is unsure about what Player 2 might say, it might be wise for her to strategise at a higher level to account for this uncertainty. She engages in a conversation, believing she is equipped with a strategy for all the situations the other player might put her into when suddenly the other player says something and she is left dumbfounded.

An example which still sticks in the memory of one of the authors after almost 20 years is the memorable line by Senator Lloyd Bentsen in his Vice-Presidential debate with Dan Quale in 1984. Quayle's strategy in the debate was to counter the perception that he was too inexperienced to have the job, and he did this by drawing similarities between his political career and former President John Kennedy's. Quayle seemed to be doing a good job in achieving his objective or winning condition, when Bentsen interrupted and said:

> Sir, I knew Jack Kennedy. I knew Jack Kennedy. And you, sir, are no Jack Kennedy.

Quayle's strategy at that point fell apart. He had no effective come back and by all accounts lost the debate handily.

The way we model this as follows. Building on [2], we take each move in a game to be a discourse which may be composed of several, even many clauses. Abstractly, we consider such discourses as sequences of basic moves, which we will be the alphabet. In a situation of incomplete information about the discourse moves, the set of moves (or the alphabet) of the Banach Mazur game being played by the players is different for the two players. Player 0 has an alphabet A (say) while Player 1 has an alphabet B such that $A \subsetneq B$. Player 0 may or may not be aware of this fact.

Thus, from the point of view of Player 0, if she is playing a Banach-Mazur game where she is unsure of the set of moves available to Player 1, it is better for her to strategise in such a way so as to account for this jump in the winning set. In other words, if Player 0's winning condition is at a level n (say) of the hierarchy, she is better off strategising for level $n+1$ given that she is unsure of Player 1's moves and given that a set at level n might undergo a jump to level $n+1$. Thus Quayle might have even won the debate had he strategiesed at a higher level expecting the unexpected.

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