Structure-Constrained Process Graphs for the Process Semantics of Regular Expressions

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Milner (1984) introduced a process semantics for regular expressions as process graphs. Unlike for the language semantics, where every regular (that is, DFA-accepted) language is the interpretation of some regular expression, there are finite process graphs that are not bisimilar to the process interpretation of any regular expression. For reasoning about graphs that are expressible by regular expressions it is desirable to have structural representations of process graphs in the image of the interpretation.

For '1-free' regular expressions, their process interpretations satisfy the structural property LEE (loop existence and elimination). But this is not in general the case for all regular expressions, as we show by examples. Yet as a remedy, we describe the possibility to recover the property LEE for a close variant of the process interpretation. For this purpose we refine the process semantics of regular expressions to yield process graphs with 1-transitions, similar to silent moves for finite-state automata.

1 Introduction

Milner [9] (1984) defined a process semantics for regular expressions as process graphs: the interpretation of 0 is deadlock, of 1 is successful termination, letters *a* are atomic actions, the operators + and \cdot stand for choice and concatenation of processes, and (unary) Kleene star (\cdot)* represents iteration with the option to terminate successfully after each pass-through. In order to disambiguate the use of regular expressions for denoting processes, Milner called them 'star expressions' in this context. Unlike for the standard language semantics, where every regular language is the interpretation of some regular expression, there are finite process graphs that are not bisimilar to the process interpretation of any star expression.¹ This phenomenon led Milner to the formulation of two questions: (R) the problem of recognizing whether a given process graph is bisimilar to one in the image of the process semantics for star expressions, and (A) whether a natural adaptation of Salomaa's complete proof system for language equivalence of regular expressions is complete for bisimilarity of process interpretations on pairs of star expressions. While (R) has been shown to be decidable in principle, so far only partial solutions have been obtained for (A).

For tackling these problems it is expedient to obtain structural representations of process graphs in the image of the interpretation. The result of Baeten, Corradini, and myself [2] that the problem (R) is decidable in principle was based on the concept of 'well-behaved (recursive) specifications' that links process graphs with star expressions. Recently in [6, 7], Wan Fokkink and I obtained a partial solution for (A) in the form of a complete proof system for '1-free' star expressions, which do not contain 1, but are formed with binary Kleene star iteration $(\cdot)^{\circledast}(\cdot)$ instead of unary iteration. For this, we defined the

¹E.g., the process graphs $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ in Ex. 2.6 on page 31 are not expressible by a star expression modulo bisimilarity.

efficiently decidable 'loop existence and elimination property (LEE)' of process graphs that holds for all process graph interpretations of 1-free star expressions, and for their bisimulation collapses.

The property LEE does unfortunately not hold for process graph interpretations $\mathscr{C}(e)$ of all star expressions *e*. However, it is the aim of this article to describe how LEE can nevertheless be made applicable, by stepping over to a variant $\underline{\mathscr{C}}(\cdot)$ of the process interpretation $\mathscr{C}(\cdot)$. In Section 3 we explain the loop existence and elimination property LEE for process graphs, and we define the concept of a 'layered LEE-witness', for short a 'LLEE-witness' for process graphs. Hereby Section 3 is an adaptation for star expressions that may contain 1 of the motivation of LLEE-witnesses in Section 3 in [6], which concerned the process semantics of '1-free star expressions'. LLEE-witnesses arise by adding natural-number labels to transitions that are subject to suitable constraints. A process graph for which a LLEE-witness exists is 'structure constrained', since it satisfies LEE (as guaranteed by the LLEE-witness) in contrast with a process graph for which no LLEE-witness exists (which then does not satisy LEE).

In Section 4 we explain examples that show that LEE does not hold in general for process interpretations of star expressions from the full class. As a remedy, we introduce process graphs with 1-transitions (similar to silent moves for finite-state automata). In Section 5 we define the variant $\underline{\mathscr{C}}(\cdot)$ of the process graph semantics $\mathscr{C}(\cdot)$ such that $\underline{\mathscr{C}}(\cdot)$ yields process graphs with 1-transitions. Then we formulate and illustrate the following two properties of the variant process graph semantics $\underline{\mathscr{C}}(\cdot)$ about its relation to $\mathscr{C}(\cdot)$, and the structure of process graphs that it defines:

- (P1) C(·) and C(·) coincide up to bisimilarity (Thm. 5.9): For every star expression e, there is a functional bisimulation from the variant process semantics C(e) of e to the process semantics C(e) of e. Therefore the process interpretation C(e) of a star expression e and its variant C(e) are bisimilar.
- (P2) $\underline{\mathscr{C}}(\cdot)$ guarantees LEE-structure (Thm. 5.14): The variant process semantics $\underline{\mathscr{C}}(e)$ of a star expression *e* satisfies the loop existence and elimination property LEE.

Section 6 is devoted to the proofs of these two properties of $\underline{\mathscr{C}}(e)$. There, we elaborate the proof of (P2). Also, we sketch the crucial steps of the proof of (P1), but refer to the report version [5] for the details.

Finally in Section 7 we briefly report on which of the steps that we developed in [6] for obtaining solutions of the problems (A) and (R) for 1-free star expressions can be extended or adapted in the direction of solutions of these problems for all star expressions, and where the remaining difficulty resides.

The idea to define structure-constrained process graphs via edge-labelings with constraints, on which LLEE-witnesses are based, originated from 'higher-order term graphs' that can be used for representing functional programs in a maximally compact, shared form (see [8, 4]). There, additional concepts (scope sets of vertices, or abstraction-prefix labelings) are used to constrain the form of term graphs. The common underlying idea with LLEE-witnesses is an enrichment of graphs that: (i) guarantees that graphs can be directly expressed by terms of some language, (ii) does not significantly hamper sharing of represented subterms, (iii) is simple enough so as to keep reasoning about graph transformations feasible.

2 Preliminaries on the process semantics of star expressions

In this section we define the process semantics of regular expressions as charts: finite labeled transition systems with initial states. We proceed by a sequence of definitions, and conclude by providing examples.

Definition 2.1. We assume, for subsequent definitions implicitly, a set *A* whose members we call *actions*. The set StExp(A) of *star expressions over (actions in) A* is defined by the following grammar:

 $e, e_1, e_2 ::= 0 | 1 | a | e_1 + e_2 | e_1 \cdot e_2 | e^*$ (where $a \in A$).

The (syntactic) star height $|e|_*$ of a star expression $e \in StExp(A)$ denotes the maximal nesting depth of stars in e via: $|0|_* := |1|_* := |a|_* := 0$, $|e_1 + e_2|_* := |e_1 \cdot e_2|_* := \max \{|e_1|_*, |e_2|_*\}$, and $|e^*|_* := 1 + |e|_*$. **Definition 2.2.** A labeled transition system (LTS) with termination and actions in A is a 4-tuple $\langle S, A, \rightarrow, \downarrow \rangle$ where S is a non-empty set of states, A is a set of action labels, $\rightarrow \subseteq S \times A \times S$ is the labeled transition relation, and $\downarrow \subseteq V$ is a set of states with immediate termination, for short, the terminating states. In such an LTS, we write $s_1 \xrightarrow{a} s_2$ for a transition $\langle s_1, a, s_2 \rangle \in \rightarrow$, and $s \downarrow$ for a terminating state $s \in \downarrow$.

Definition 2.3. The transition system specification (TSS) $\mathscr{T}(A)$ is defined by the axioms and rules:

$$\frac{e_{i}\downarrow}{(e_{1}+e_{2})\downarrow} \stackrel{(i \in \{1,2\})}{(e_{1}+e_{2})\downarrow} \quad \frac{e_{1}\downarrow}{(e_{1}\cdot e_{2})\downarrow} \quad \overline{(e^{*})\downarrow}$$

$$\frac{e_{i}\stackrel{a}{\rightarrow} e_{i}'}{e_{1}+e_{2}\stackrel{a}{\rightarrow} e_{i}'} \stackrel{(i \in \{1,2\})}{(i \in \{1,2\})} \quad \frac{e_{1}\stackrel{a}{\rightarrow} e_{1}'}{e_{1}\cdot e_{2}\stackrel{a}{\rightarrow} e_{1}'} \quad \frac{e_{1}\downarrow}{e_{1}\cdot e_{2}\stackrel{a}{\rightarrow} e_{2}'} \quad \frac{e\stackrel{a}{\rightarrow} e_{2}'}{e_{1}\cdot e_{2}\stackrel{a}{\rightarrow} e_{2}'}$$

If $e \xrightarrow{a} e'$ is derivable in $\mathscr{T}(A)$, for $e, e' \in StExp(A)$, and $a \in A$, then we say that e' is a *derivative* of e. If $e \downarrow$ is derivable in $\mathscr{T}(A)$, for $e \in StExp(A)$, then we say that e permits immediate termination. If $e \downarrow$ is not derivable in $\mathscr{T}(A)$, then we write $e \ddagger$.

The TSS $\mathscr{T}(A)$ defines the process semantics for star expressions in StExp(A) in the form of the *star* expressions LTS $\mathscr{L}(StExp(A)) := \mathscr{L}_{\mathscr{T}(A)}$, where $\mathscr{L}_{\mathscr{T}(A)} = \langle StExp(A), A, \rightarrow, \downarrow \rangle$ is the LTS generated by $\mathscr{T}(A)$, that is, its set $\rightarrow \subseteq StExp(A) \times A \times StExp(A)$ of transitions, and its set $\downarrow \subseteq StExp(A)$ of vertices with the immediate-termination property are defined via derivations in $\mathscr{T}(A)$ in the natural way.

For every set $S \subseteq StExp(A)$ we denote by $\mathscr{L}(S)$ the *S*-generated sub-LTS $\langle V_S, A, \rightarrow_S, \downarrow_S \rangle$ of the star expressions LTS $\mathscr{L}(StExp(A))$, that is, the sub-LTS whose states are those in *S* together with all star expressions that are reachable from ones in *S* via paths that follow transitions of $\mathscr{L}(StExp(A))$.

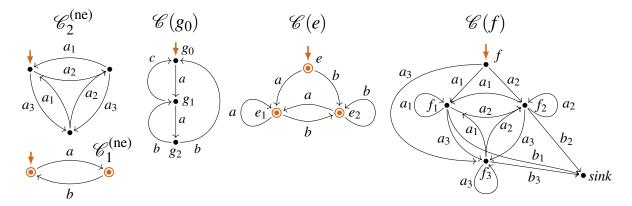
Definition 2.4. A *chart* is a 5-tuple $\mathscr{C} = \langle V, A, v_s, \rightarrow, \downarrow \rangle$ such that $\langle V, A, \rightarrow, \downarrow \rangle$ is an LTS, which we call the *LTS underlying* \mathscr{C} , with a <u>finite</u> set *V* of states, which we call *vertices*, and that is rooted by a specified *start vertex* $v_s \in V$; we call $\rightarrow \subseteq V \times A \times V$ the set of *labeled transitions* of \mathscr{C} , and $\downarrow \subseteq V$ the set the vertices with immediate termination of \mathscr{C} .

Definition 2.5. The *chart interpretation* $\mathscr{C}(e) = \langle V(e), A, e, \rightarrow, \downarrow \rangle$ of a star expression $e \in StExp(A)$ is the $\{e\}$ -generated sub-LTS $\mathscr{L}(\{e\}) = \langle V_{\{e\}}, A, \rightarrow_{\{e\}}, \downarrow_{\{e\}} \rangle$ of $\mathscr{L}(StExp(A))$.²

Example 2.6. In the chart illustrations below and later, we indicate the start vertex by a brown arrow \rightarrow , and the property of a vertex *v* to permit immediate termination by emphasizing *v* in brown as \bigcirc including a boldface ring. Each of the vertices of the charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}(e)$ below permits immediate termination, but none of the vertices of the other charts does. The charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ are not expressible by the process interpretation modulo bisimilarity, as shown by Bosscher [3] and Milner [9]. That $\mathscr{C}_2^{(ne)}$ is not expressible, Milner proved by observing the absence of a 'loop behaviour'. That concept has inspired the stronger concept of 'loop chart' in Def. 3.1 below. For the weaker result that $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ are not expressible by 1-free star expressions please see Remark 3.6.

The chart $\mathscr{C}(g_0)$ middle left below is the interpretation of the star expression $g_0 = ((1 \cdot a) \cdot g) \cdot 0$ where $g = (c \cdot a + a \cdot (b + b \cdot a))^*$, and with $g_1 = (1 \cdot g) \cdot 0$, and $g_2 = ((1 \cdot (b + b \cdot a)) \cdot g) \cdot 0$ as remaining vertices. The chart $\mathscr{C}(e)$ is the interpretation of $e = (a^* \cdot b^*)^*$ with $e_1 = ((1 \cdot a^*) \cdot b^*) \cdot e$, and $e_2 = (1 \cdot b^*) \cdot e$. With $f_0 = a_1 \cdot (1 + b_1 \cdot 0) + a_2 \cdot (1 + b_2 \cdot 0) + a_3 \cdot (1 + b_3 \cdot 0)$, the chart $\mathscr{C}(f)$ is the interpretation of $f = f_0^* \cdot 0$ with $f_i = (1 \cdot (1 + b_i \cdot 0) \cdot f_0^*) \cdot 0$ (for $i \in \{1, 2, 3\}$), and $sink = ((1 \cdot 0) \cdot f_0^*) \cdot 0$.

²That this generated sub-LTS is finite, and so that $\mathscr{C}(e)$ is indeed a chart, follows from Antimirov's result in [1] that every regular expression has only finitely many iterated 'partial derivatives', which coincide with repeated derivatives from Def. 2.3.



The chart interpretations $\mathscr{C}(e)$ and $\mathscr{C}(f)$, which will be used later, have been constructed as expressible variants of the not expressible charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$. In particular, $\mathscr{C}(e)$ contains $\mathscr{C}_1^{(ne)}$ as a subchart, and $\mathscr{C}(f)$ contains $\mathscr{C}_2^{(ne)}$ as a subchart (a 'subchart' arises by taking a part of a chart, and picking a start vertex). We finally note that all of these charts with the exception of $\mathscr{C}(e)$ are bisimulation collapses.

Definition 2.7. For $i \in \{1,2\}$ we consider the LTSs $\mathscr{L}_i = \langle S_i, A, \rightarrow_i, \downarrow_i \rangle$. By a *bisimulation between* \mathscr{L}_1 and \mathscr{L}_2 we mean a binary relation $B \subseteq S_1 \times S_2$ with the properties that it is non-empty, that is, $B \neq \emptyset$, and that for every $\langle s_1, s_2 \rangle \in B$ the following three conditions hold:

(forth) $\forall s'_1 \in S_1 \forall a \in A\left(s_1 \xrightarrow{a} s'_1 \implies \exists s'_2 \in S_2\left(s_2 \xrightarrow{a} s'_2 \land \langle s'_1, s'_2 \rangle \in B\right)\right),$ (back) $\forall s'_2 \in S_2 \forall a \in A\left(\left(\exists s'_1 \in S_1\left(s_1 \xrightarrow{a} s'_1 \land \langle s'_1, s'_2 \rangle \in B\right)\right) \iff s_2 \xrightarrow{a} s'_2 s'_2\right),$

(termination) $s_1 \downarrow_1 \iff s_2 \downarrow_2$.

For a partial function $f: S_1 \to S_2$ we say that f defines a bisimulation between \mathcal{L}_1 and \mathcal{L}_2 if its graph $\{\langle v, f(v) \rangle \mid v \in S_1\}$ is a bisimulation between \mathcal{L}_1 and \mathcal{L}_2 . We call this graph a *functional* bisimulation.

Definition 2.8 (bisimulation between charts). For $i \in \{1,2\}$ we consider the charts $\mathscr{C}_i = \langle V_i, A, v_{s,i}, \rightarrow_i, \downarrow_i \rangle$. By a *bisimulation between* \mathscr{C}_1 and \mathscr{C}_2 we mean a binary relation $B \subseteq V_1 \times V_2$ such that $\langle v_{s,1}, v_{s,2} \rangle \in B$ (*B* relates the start vertices of \mathscr{C}_1 and \mathscr{C}_2), and *B* is a bisimulation between the underlying LTSs.

We denote by $\mathscr{C}_1 \nleftrightarrow \mathscr{C}_2$, and say that \mathscr{C}_1 and \mathscr{C}_2 are *bisimilar*, if there is a bisimulation between \mathscr{C}_1 and \mathscr{C}_2 . By $\mathscr{C}_1 \rightrightarrows \mathscr{C}_2$ we denote the stronger statement that there is a partial function $f : V_1 \rightharpoonup V_2$ whose graph $\{\langle v, f(v) \rangle \mid v \in V_1\}$ is a bisimulation between \mathscr{C}_1 and \mathscr{C}_2 .

Each of four charts $\mathscr{C}_1^{(ne)}$, $\mathscr{C}_2^{(ne)}$, $\mathscr{C}(g_0)$, and $\mathscr{C}(f)$ in Ex. 2.6 is a 'bisimulation collapse': by that we mean a chart for which no two subcharts that are induced by different vertices are bisimilar. But that does not hold for $\mathscr{C}(e)$ in which any two subcharts that are induced by different vertices are bisimilar.

3 Loop existence and elimination

The chart translation $\mathscr{C}(g_0)$ of g_0 as in Ex. 2.6 satisfies the 'loop existence and elimination' property LEE that we will explain in this section. For this purpose we summarize Section 3 in [6], and in doing so we adapt the concepts defined there from 1-free star expressions to the full class of star expressions as defined in Section 2. The property LEE is defined by a dynamic elimination procedure that analyses the structure of a chart by peeling off 'loop subcharts'. Such subcharts capture, within the chart interpretation of a star expression e, the behavior of the iteration of f within innermost subterms f^* in e.

Definition 3.1. A chart $\mathscr{L} = \langle V, A, v_s, \rightarrow, \downarrow \rangle$ is called a *loop chart* if:

- (L1) There is an infinite path from the start vertex v_s .
- (L2) Every infinite path from v_s returns to v_s after a positive number of transitions (and so visits v_s infinitely often).
- (L3) Immediate termination is only permitted at the start vertex, that is, $\downarrow \subseteq \{v_s\}$.

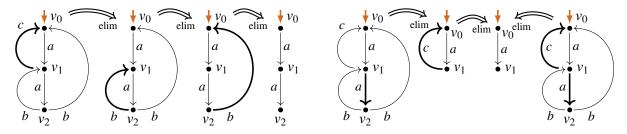
We call the transitions from v_s *loop-entry transitions*, and all other transitions *loop-body transitions*. A loop chart \mathscr{L} is a *loop subchart of* a chart \mathscr{C} if it is the subchart of \mathscr{C} rooted at some vertex $v \in V$ that is generated, for a nonempty set U of transitions of \mathscr{C} from v, by all paths that start with a transition in U and continue onward until v is reached again (so the transitions in U are the loop-entry transitions of \mathscr{L}).

Both of the not expressible charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ in Ex. 2.6 are not loop charts: $\mathscr{C}_1^{(ne)}$ violates (L3), and $\mathscr{C}_2^{(ne)}$ violates (L2). Moreover, none of these charts contains a loop subchart. The chart $\mathscr{C}(g_0)$ in Ex. 2.6 is not a loop chart either, as it violates (L2). But we will see that $\mathscr{C}(g_0)$ has loop subcharts.

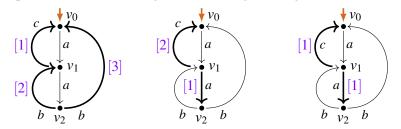
Let \mathscr{L} be a loop subchart of a chart \mathscr{C} . The result of *eliminating* \mathscr{L} from \mathscr{C} arises by removing all loop-entry transitions of \mathscr{L} from \mathscr{C} , and then removing all vertices and transitions that become unreachable. A chart \mathscr{C} has the *loop existence and elimination property* (*LEE*) if the procedure, started on \mathscr{C} , of repeated eliminations of loop subcharts results in a chart that does not have an infinite path.

For the not expressible charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ in Ex. 2.6 the procedure stops immediately, as they do not contain loop subcharts. As both of them have infinite paths, it follows that they do not satisfy LEE.

Now we consider (see below) three runs of the elimination procedure for the chart $\mathscr{C}(g_0)$ in Ex. 2.6. The loop-entry transitions of loop subcharts that are removed in each step are marked in bold. Each run witnesses that $\mathscr{C}(g_0)$ satisfies LEE. Note that loop elimination does not yield a unique result.³



Runs can be recorded, in the original chart, by attaching a marking label to transitions that get removed in the elimination procedure. That label is the sequence number of the corresponding elimination step. For the three runs of loop elimination above we get the following marking labeled versions of C, respectively:



Since all three runs were successful (as they yield charts without infinite paths), these recordings (markinglabeled charts) can be viewed as 'LEE-witnesses'. We now will define the concept of a 'layered LEE-witness' (LLEE-witness), i.e., a LEE-witness with the added constraint that in the recorded run of the loop elimination procedure it never happens that a loop-entry transition is removed from within the body of a previously removed loop subchart. This refined concept has simpler properties, but is equally powerful.

³Confluence can be shown if a pruning operation is added that permits to drop transitions to deadlocking vertices.

Definition 3.2. An *entry/body-labeling of a chart* $\mathscr{C} = \langle V, A, v_s, \rightarrow, \downarrow \rangle$ is a chart $\widehat{\mathscr{C}} = \langle V, A \times \mathbb{N}, v_s, \hat{\rightarrow}, \downarrow \rangle$ where $\widehat{\rightarrow} \subseteq A \times \mathbb{N}$ arises from \rightarrow by adding, for each transition $\tau = \langle v_1, a, v_2 \rangle \in \rightarrow$, to the action label *a* of τ a *marking label* $n \in \mathbb{N}$, yielding $\widehat{\tau} = \langle v_1, \langle a, n \rangle, v_2 \rangle \in \widehat{\rightarrow}$. In an entry/body-labeling we call transitions with marking label 0 *body transitions*, and transitions with marking labels in \mathbb{N}^+ *entry transitions*.

By an *entry/body-labeling of an LTS* $\mathscr{L} = \langle S, A, \rightarrow, \downarrow \rangle$ we mean an LTS $\widehat{\mathscr{L}} = \langle S, A \times \mathbb{N}, \hat{\rightarrow}, \downarrow \rangle$ where $\hat{\rightarrow} \subseteq S \times (A \times \mathbb{N}) \times S$ arises from \rightarrow by adding marking labels in \mathbb{N} to the action labels of transitions.

We define the following designations and concepts for LTSs, but will use them also for charts. Let $\widehat{\mathscr{L}}$ be an entry/body-labeling of \mathscr{L} , and let v and w be vertices of \mathscr{L} and $\widehat{\mathscr{L}}$. We denote by $v \to_{bo} w$ that there is a body transition $v \xrightarrow{\langle a, 0 \rangle} w$ in $\widehat{\mathscr{L}}$ for some $a \in A$, and by $v \to_{[n]} w$, for $n \in \mathbb{N}^+$ that there is an entry-transition $v \xrightarrow{\langle a, n \rangle} w$ in $\widehat{\mathscr{L}}$ for some $a \in A$. By the set $E(\widehat{\mathscr{L}})$ of *entry-transition identifiers* we denote the set of pairs $\langle v, n \rangle \in V \times \mathbb{N}^+$ such that an entry-transition $\to_{[n]}$ departs from v in $\widehat{\mathscr{L}}$. For $\langle v, n \rangle \in E(\widehat{\mathscr{L}})$, we define by $\mathscr{C}_{\widehat{\mathscr{L}}}(v, n)$ the subchart of \mathscr{L} with start vertex v_s that consists of the vertices and transitions which occur on paths in \mathscr{L} as follows: any path that starts with a $\to_{[n]}$ entry-transition from v, continues with body transitions only (thus does not cross another entry-transition), and halts immediately if v is revisited.

The three recordings obtained above of the loop elimination procedure for the chart $\mathscr{C}(g_0)$ in Ex. 2.6 indicate entry/body-labelings by signaling the entry-transitions but neglecting body-step labels 0.

Definition 3.3. Let $\mathscr{L} = \langle V, A, \rightarrow, \downarrow \rangle$ be an LTS. A LLEE-*witness* (a *layered* LEE-*witness*) of \mathscr{L} is an entry/body-labeling $\widehat{\mathscr{L}}$ of \mathscr{L} that satisfies the following three properties:

- (W1) Body-step termination: There is no infinite path of \rightarrow_{bo} transitions in \mathscr{L} .
- (W2) Loop condition: For all $\langle s,n\rangle \in E(\widehat{\mathscr{L}}), \ \mathscr{C}_{\widehat{\mathscr{L}}}(s,n)$ is a loop chart.
- (W3) Layeredness: For all $\langle s, n \rangle \in E(\widehat{\mathscr{L}})$, if an entry-transition $t \to_{[m]} t'$ departs from a state $t \neq s$ of $\mathscr{C}_{\widehat{\mathscr{L}}}(s,n)$, then its marking label *m* satisfies m < n.

The condition (W2) justifies to call an entry-transition in a LLEE-witness a *loop-entry transition*. For a loop-entry transition $\rightarrow_{[m]}$ with $m \in \mathbb{N}^+$, we call *m* its *loop level*.

For a chart $\mathscr{C} = \langle V, A, v_s, \rightarrow, \downarrow \rangle$, we define a LLEE-*witness* $\widehat{\mathscr{C}}$ analogously as an entry/body-labeling $\widehat{\mathscr{C}}$ of \mathscr{C} with the properties (W1), (W2), and (W3) with \mathscr{C} and $\widehat{\mathscr{C}}$ for \mathscr{L} and $\widehat{\mathscr{L}}$, respectively.

Example 3.4. The three entry/body-labelings of the chart $\mathscr{C}(g_0)$ in Ex. 2.6 that we have obtained as recordings of runs of the loop elimination procedure are LLEE-witnesses of $\mathscr{C}(g_0)$, as is easy to verify.

Proposition 3.5. If a chart C has a LLEE-witness, then it satisfies LEE.

Proof. Let $\widehat{\mathscr{C}}$ be a LLEE-witness of a chart \mathscr{C} . Repeatedly pick an entry-transition identifier $\langle v, n \rangle \in E(\widehat{\mathscr{C}})$ with $n \in \mathbb{N}^+$ minimal, remove the loop subchart that is generated by loop-entry transitions of level n from v (it is indeed a loop by condition (W2) on $\widehat{\mathscr{C}}$, noting that minimality of n and condition (W3) on $\widehat{\mathscr{C}}$ ensure the absence of departing loop-entry transitions of lower level), and perform garbage collection. Eventually the part of \mathscr{C} that is reachable by body transitions from the start vertex is obtained. This subchart does not have an infinite path due to condition (W1) on $\widehat{\mathscr{C}}$. Therefore \mathscr{C} satisfies LEE.

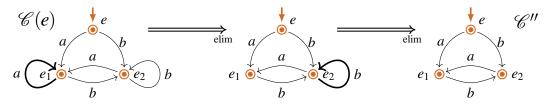
The condition (W2) on a LLEE-witness $\widehat{\mathscr{C}}$ of a chart \mathscr{C} requires the loop structure defined by $\widehat{\mathscr{C}}$ to be hierarchical. This permits to extract a star expression \tilde{e} from $\widehat{\mathscr{C}}$ (defined in [6, 7]) that expresses \mathscr{C} in the sense that $\mathscr{C}(\tilde{e}) \hookrightarrow \mathscr{C}$ holds, intuitively by unfolding the underlying chart \mathscr{C} to the syntax tree of \tilde{e} .

Remark 3.6. In [6, 7] we established a connection between charts that have a LLEE-witness (and hence satisfy LEE) and charts that are expressible by 1-free star expressions (that is, star expressions without 1,

and with binary star iteration instead of unary star iteration). By saying that a chart \mathscr{C} 'is expressible' we mean here that \mathscr{C} is bisimilar to the chart interpretation $\mathscr{C}(\tilde{e})$ of some star expression \tilde{e} . Now Cor. 6.10 in [6, 7] states that if a chart is expressible by a 1-free star expression then its bisimulation collapse has a LLEE-witness, and thus satisfies LEE. This statement entails that neither of the charts $\mathscr{C}_1^{(ne)}$ and $\mathscr{C}_2^{(ne)}$ in Ex. 2.6 is expressible by a 1-free star expression, because both are bisimulation collapses, and neither of the satisfies LEE, as we have already observed above.

4 LEE may fail for process interpretations of star expressions

The chart interpretations $\mathscr{C}(e)$ of e, and $\mathscr{C}(f)$ of f in Ex. 2.6 do not satisfy LEE, contrasting with $\mathscr{C}(g_0)$. For $\mathscr{C}(e)$ we find the following run of the loop elimination procedure that successively eliminates the two loop subcharts that are induced by the cycling transitions at e_1 and e_2 :



The resulting chart \mathscr{C}'' does not contain loop subcharts any more, because taking, for example, a transition from e_1 to e_2 as an entry-transition does not yield a loop subchart, because in the induced subchart immediate termination is not only possible at the start vertex e_1 but also in the body vertex e_2 , in contradiction to (L3). But while \mathscr{C}'' does not contain a loop subchart any more, it still has an infinite trace. Therefore it follows that $\mathscr{C}(e)$ does not satisfy LEE.

In order to see that $\mathscr{C}(f)$ does not satisfy LEE, we can consider a run of the loop elimination procedure that successively removes the cyclic transitions at f_1 , f_2 , and f_3 . After these removals a variant of the not expressible chart $\mathscr{C}_2^{(ne)}$ is obtained that still describes an infinite behavior, but that does not contain any loop subchart. The latter can be argued analogously as for $\mathscr{C}_2^{(ne)}$, namely that for all choices of entry-transitions between f_1 , f_2 , and f_3 the loop condition (L2) fails. We conclude that $\mathscr{C}(f)$ does not satisfy LEE.

The reason for this failure of LEE is that, while the syntax trees of star expressions can provide a nested loop-chart structure, this is not guaranteed by the specific form of the TSS \mathscr{T} . Execution of an iteration g^* in an expression $g^* \cdot h$ leads eventually, in case that termination is reachable in g, to an iterated derivative $(1 \cdot g^*) \cdot h$. Also, as in the examples above, an iterated derivative $(\tilde{g} \cdot g^*) \cdot h$ with $\tilde{g} \downarrow$ may be reached. In these cases, continued execution will bypass the initial term $g^* \cdot h$, and either proceed with another execution of the iteration to $(g' \cdot g^*) \cdot h$, where g' is a derivative of g, or take a step into the exit to h', where h' is a derivative of h. In both cases the execution does not return to the initial term g^* of the execution, as would be required for a loop subchart at g^* to arise in accordance with loop condition (L2).

5 Recovering LEE for a variant definition of the process semantics

A remedy for the frequent failure of LEE for the chart translation of star expressions can consist in the use of '1-transitions'. Such transitions may be used to create a back-link to an expression $g^* \cdot h$ from an iterated derivative $(\tilde{g} \cdot g^*) \cdot h$ with $\tilde{g} \downarrow$ (where \tilde{g} is an iterated derivative of g) that is reached by a descent of the execution into the body of g. This requires an adapted refinement of the TSS \mathscr{T} from page 31.

While different such refinements are conceivable, our choice is to introduce 1-transitions most sparingly, making sure that every 1-transition can be construed as a backlink in an accompanying LLEE-witness.

In particular we want to create transition rules that facilitate a back-link to an expression g^* after the execution has descended into g reaching $\tilde{g} \cdot g^*$ with $\tilde{g} \downarrow$. In order to distinguish a concatenation expression $\tilde{g} \cdot g^*$ that arises from the descent of the execution into an iteration g^* from other concatenation expressions we introduce a variant operation *. The rules of the refined TSS should guarantee that in the example the reached iterated derivative of g^* is a 'stacked star expression' $G * g^*$ where G is itself a stacked star expression that denotes an iterated derivative of g. If now G is also a star expression \tilde{g} with $\tilde{g} \downarrow$, then the expression $G * g^*$ of the form $\tilde{g} * g^*$ should permit a 1-transition that returns to g^* .

This intuition guided the definition of the rules of the TSS $\underline{\mathscr{T}}^{(*)}$ in Def. 5.3 below, starting from the adaptation of the rule for steps from iterations e^* , and the rule that creates 1-transition backlinks to iterations e_2^* from stacked expressions $E_1 * e_2^*$ with $E_1 \downarrow$. The 'stacked product' * has the following features: $E * e^*$ never permits immediate termination; for defining transitions it behaves similarly as concatenation \cdot except that a transition from $E * e^*$ into e^* when E permits immediate termination now requires a 1-transition to e^* first. The formulation of these rules of $\underline{\mathscr{T}}^{(*)}$ led to the tailor-made set of stacked star expressions as defined below.

Definition 5.1. Let A be a set whose members we call *actions*. The set $StExp^{(*)}(A)$ of *stacked star expressions over (actions in)* A is defined by the following grammar:

$$E ::= e \mid E \cdot e \mid E * e^*$$
 (where $e \in StExp(A)$)

Note that the set StExp(A) of star expressions would arise again if the clause $E * e^*$ were dropped.

The star height $|E|_*$ of stacked star expressions E is defined by adding $|E \cdot e|_* := \max\{|E|_* + |e|_*\}$, and $|E * e^*|_* := \max\{|E|_* + |e^*|_*\}$ to the defining clauses for star height of star expressions.

The projection function π : $StExp^{(*)}(A) \rightarrow StExp(A)$ is defined by interpreting * as \cdot by the clauses: $\pi(E \cdot e) := \pi(E) \cdot e, \ \pi(E * e^*) := \pi(E) \cdot e^*, \ \text{and} \ \pi(e) := e, \ \text{for all} \ E \in StExp^{(*)}(A), \ \text{and} \ e \in StExp(A).$

Definition 5.2. By a *labeled transition system with termination, actions in A and empty steps (a 1-LTS)* we mean a 5-tuple $\langle S, A, 1, \rightarrow, \downarrow \rangle$ where *S* is a non-empty set of *states, A* is a set of (*proper*) *action labels,* $1 \notin A$ is the specified *empty step label,* $\rightarrow \subseteq S \times \underline{A} \times S$ is the *labeled transition relation,* where $\underline{A} := A \cup \{1\}$ is the set of action labels including 1, and $\downarrow \subseteq V$ is a set of *states with immediate termination.* Note that then $\langle S, \underline{A}, \rightarrow, \downarrow \rangle$ is an LTS. In such a 1-LTS, we call a transition in $\rightarrow \cap (S \times A \times S)$ (labeled by a *proper action* in *A*) a *proper transition,* and a transition in $\rightarrow \cap (S \times \{1\} \times S)$ (labeled by the *empty-step symbol* 1) a 1-*transition*. Reserving non-underlined action labels like *a*, *b*,... for proper actions, we use underlined action label symbols like \underline{a} for actions labels in the set \underline{A} that includes the label 1.

Definition 5.3. The transition system specification $\underline{\mathscr{T}}^{(*)}(A)$ has the following axioms and rules, where $1 \notin A$ is an additional label (for representing empty steps), $a \in A$, $\underline{a} \in \underline{A} := A \cup \{1\}$, stacked star expressions $E_1, E_2, E'_1, E'_2, E' \in \underline{\mathscr{T}}^{(*)}(A)$, and star expressions $e_1, e_2, e_2^*, e^* \in StExp(A)$ (here and below we highlight in red transitions that may involve 1-transitions):

$$\frac{a \stackrel{a}{\rightarrow} 1}{a \stackrel{a}{\rightarrow} 1} = \frac{\begin{array}{c} \frac{e_{i\downarrow}}{1\downarrow} & (i \in \{1,2\}) \\ \hline e_{i} \stackrel{a}{\rightarrow} E'_{i} \\ \hline e_{1} + e_{2} \stackrel{a}{\rightarrow} E'_{i} \end{array}}{(i \in \{1,2\})} (i \in \{1,2\}) \\ \hline \begin{array}{c} E_{1} \stackrel{a}{\rightarrow} E'_{1} \\ \hline E_{1} \cdot e_{2} \stackrel{a}{\rightarrow} E'_{1} \\ \hline E_{1} \cdot e_{2} \stackrel{a}{\rightarrow} E'_{1} \\ \hline \end{array} \\ \hline \begin{array}{c} e_{1\downarrow} & e_{2} \stackrel{a}{\rightarrow} E'_{2} \\ \hline e_{1} \cdot e_{2} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{1} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{1} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \begin{array}{c} e^{*} \stackrel{a}{\rightarrow} E'_{2} \\ \hline \end{array} \\ \hline \end{array}$$

If $E \xrightarrow{a} E'$ is derivable in $\underline{\mathscr{T}}^{(*)}(A)$, for $E, E' \in StExp(A)$, and $\underline{a} \in \underline{A}$, then we say that E' is a *subderivative* of E. The TSS $\underline{\mathscr{T}}^{(*)}(A)$ defines the variant process semantics for stacked star expressions in $StExp^{(*)}(A)$ as the *stacked star expressions* 1-*LTS* $\underline{\mathscr{L}}(StExp^{(*)}(A)) := \underline{\mathscr{L}}_{\underline{\mathscr{T}}^{(*)}(A)}$, where $\underline{\mathscr{L}}_{\underline{\mathscr{T}}^{(*)}(A)} = \langle StExp^{(*)}(A), A, \rightarrow, \downarrow \rangle$ is the *1*-*LTS generated by* $\underline{\mathscr{T}}^{(*)}(A)$, that is, its transitions $\rightarrow \subseteq StExp^{(*)}(A) \times \underline{A} \times StExp^{(*)}(A)$, and its immediately terminating vertices $\downarrow \subseteq StExp^{(*)}(A)$ are defined via derivations in $\mathcal{T}(A)$ in the natural way. For sets $S \subseteq StExp^{(*)}(A)$, *S*-generated sub-1-LTSs are defined analogously as for $\mathscr{L}(StExp(A))$.

Definition 5.4. A 1-*chart* is a (rooted) 1-LTS $\langle V, A, 1, v_s, \rightarrow, \downarrow \rangle$ such that $\langle V, A, 1, \rightarrow, \downarrow \rangle$ is a 1-LTS whose states we refer to as *vertices*, and where $v_s \in V$ is called the *start vertex* of the 1-chart.

Definition 5.5. The 1-*chart interpretation* $\underline{\mathscr{C}}(e) = \langle V(e), A, \mathbf{1}, e, \rightarrow_e, \downarrow_e \rangle$ of a star expression $e \in StExp(A)$ is the *e*-rooted version of the $\{e\}$ -generated sub-1-LTS $\underline{\mathscr{L}}(\{e\}) = \langle V_{\{e\}}, A, \mathbf{1}, \rightarrow_{\{e\}}, \downarrow_{\{e\}} \rangle$ of $\underline{\mathscr{L}}(StExp(A))$.

In order to link the 1-LTS $\mathcal{L}(StExp^{(*)}(A))$ to the LTS $\mathcal{L}(StExp(A))$, we need to take account of the semantics of 1-transitions as empty steps (see Vrancken [10]). For this, we define 'induced transitions' $\stackrel{(\cdot)}{\rightarrow}$, and 'induced termination' $\downarrow^{(1)}$ as follows: $E \xrightarrow{(a]} E'$ holds if there is a sequence of 1-transitions from E to some \widetilde{E} from which there is an *a*-transition to E' (the asymmetric notation $\xrightarrow{(\cdot)}$ is intended to reflect this asymmetry), and $E \downarrow^{(1)}$ holds if there is a sequence of 1-transitions from E to some \widetilde{E} with $\widetilde{E} \downarrow$.

Definition 5.6. Let $\underline{\mathscr{L}} = \langle S, A, 1, \rightarrow, \downarrow \rangle$ be a 1-LTS. By the *induced LTS of* $\underline{\mathscr{L}}$, and the *LTS induced by* $\underline{\mathscr{L}}$, we mean the LTS $\underline{\mathscr{L}}_{(\cdot]} = \langle S, A, \stackrel{(\cdot]}{\rightarrow}, \downarrow^{(1)} \rangle$ where $\stackrel{(\cdot]}{\rightarrow} \subseteq S \times \underline{A} \times S$ is the *induced transition relation*, and $\downarrow^{(1)} \subseteq V$ is the set of vertices with *induced termination* that are defined as follows, for all $s, s' \in S$ and $a \in A$:

- (ind-1) $s \xrightarrow{(a)} s'$ holds if $s = s_0 \xrightarrow{1} s_1 \xrightarrow{1} \dots \xrightarrow{1} s_n \xrightarrow{a} s'$, for some $s_0, \dots, s_n \in V$ and $n \in \mathbb{N}$ (we then say that there is an *induced transition* between $s, s' \in S$ with respect to \mathcal{L}),
- (ind-2) $s\downarrow^{(1)}$ holds if $s = s_0 \xrightarrow{1} s_1 \xrightarrow{1} \dots \xrightarrow{1} s_n \wedge s_n \downarrow^{(1)}$, for some $s_0, \dots, s_n \in V$ and $n \in \mathbb{N}$ (then we say that *s* has induced termination with respect to $\underline{\mathscr{L}}$).

Definition 5.7. The *induced chart* $\underline{\mathscr{C}}_{(\cdot]} = \langle V, A, v_s, \stackrel{(\cdot)}{\to}, \downarrow^{(1)} \rangle$ of a 1-chart $\underline{\mathscr{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow \rangle$ is defined analogously as the induced LTS of a 1-LTS in Def. 5.6 with the induced transition relation $\stackrel{(\cdot)}{\to}$ defined analogously to (ind-1), and the set of vertices $\downarrow^{(1)}$ with induced termination defined analogously to (ind-2).

Lemma 5.8. The projection function π defines a bisimulation between the induced LTS $\mathcal{L}(StExp^{(*)}(A))_{(\cdot)}$ of the stacked star expressions 1-LTS StExp^{(*)}(A) and the star expressions LTS $\mathcal{L}(StExp(A))$.

With this lemma, the proof of which we outline in Section 6, we will be able to prove the following connection between the chart interpretation and the 1-chart interpretation of a star expression.

Theorem 5.9. $\underline{\mathscr{C}}(e)_{(\cdot]} \Rightarrow \mathscr{C}(e)$ holds for all $e \in StExp(A)$, that is, there is a functional bisimulation from the induced chart of the 1-chart interpretation of a star expression e to the chart interpretation of e.

For the construction of LLEE-witnesses for the 1-chart interpretation $\mathcal{C}(\cdot)$ we need to distinguish 'normed⁺' stacked star expressions that permit an induced-transition path of <u>positive length</u> to an expression with induced termination from those that do not enable such a path. This property slightly strengthens normedness of expressions, which means the existence of a path to an expression with immediate termination (or equally, an <u>arbitrary-length</u> induced-transition path to an expression with induced termination).

Definition 5.10. Let $E \in StExp^{(*)}(A)$. We say that E is *normed*⁺ (and E is *normed*) if there is $E' \in StExp^{(*)}(A)$ such that $E \xrightarrow{(\cdot)} E'$ and $E' \downarrow^{(1)}$ in $\mathcal{L}(StExp(A))_{(\cdot)}$ (resp., $E \to E'$ and $E' \downarrow$ in $\mathcal{L}(StExp(A))$).

These properties permit inductive definitions, and therefore they are easily decidable. Also, a stacked star expression is normed⁺ if and only if it enables a transition to a normed stacked star expression.

Now we define, similarly as we have done so for 1-free star expressions in [6, 7], a refinement of the TSS $\underline{\mathscr{T}}^{(*)}$ into a TSS that will supply entry/body-labelings for LLEE-witnesses, by adding marking labels to the rules of $\underline{\mathscr{T}}^{(*)}$. In particular, body labels are added to transitions that cannot return to their source expression. The rule for transitions from an iteration e^* is split into the case in which e is normed⁺ or not. Only if e is normed⁺ can e^* return to itself after a positive number of steps, and then a (loop-) entry-transition with the star height $|e^*|_*$ of e^* as its level is created; otherwise a body label is introduced.

Definition 5.11. The TSS $\underline{\widehat{\mathscr{T}}}^{(*)}(A)$ has the following rules, where $l \in \{bo\} \cup \{[n] \mid n \in \mathbb{N}^+\}$:

$$\begin{array}{cccc} \hline \hline 1\downarrow & \hline \hline e_i\downarrow & (i \in \{1,2\}) & \hline e_1\downarrow & e_2\downarrow & \hline \hline (e^*)\downarrow \\ \hline \hline \hline 1\downarrow & \hline \hline (e_1+e_2)\downarrow & (i \in \{1,2\}) & \hline \hline e_1\downarrow & e_2\downarrow & \hline \hline (e^*)\downarrow \\ \hline \hline \hline a \xrightarrow{a}_{bo} 1 & \hline e_i\xrightarrow{a}_l E'_i & (i \in \{1,2\}) & \hline \hline E_1\xrightarrow{a}_l E'_1 & e_2\xrightarrow{a}_l E'_2 \\ \hline \hline e_1+e_2\xrightarrow{a}_{bo} E'_i & (i \in \{1,2\}) & \hline \hline E_1\cdot e_2\xrightarrow{a}_l E'_1 \cdot e_2 & \hline e_1\downarrow & e_2\xrightarrow{a}_l E'_2 \\ \hline \hline e_1\cdot e_2\xrightarrow{a}_{bo} E'_i & (i \in \{1,2\}) & \hline \hline E_1\cdot e_2\xrightarrow{a}_l E'_1 \cdot e_2 & \hline e_1 \cdot e_2\xrightarrow{a}_{bo} E'_2 \\ \hline \hline e^*\xrightarrow{a}_{[le^*]*]} E' * e^* & \hline e^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline E_1 \times e_2^*\xrightarrow{a}_l E'_1 & e_1\downarrow \\ \hline \hline e_1 + e_2\xrightarrow{a}_{bo} E' * e^* & \hline \hline e^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline \hline e_1 \times e_2^*\xrightarrow{a}_l E'_1 & e_1\downarrow \\ \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} e_2^* \\ \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' \\ \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' * e^* & \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo} E' \\ \hline \hline e_1 + e_2^*\xrightarrow{a}_{bo$$

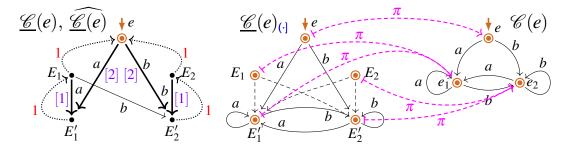
The entry/body-labeling $\widehat{\mathscr{L}}(StExp^{(*)}(A)) = \langle StExp(A), \underline{A}, \rightarrow, \downarrow \rangle$ of the stacked star expressions 1-LTS $\underline{\mathscr{L}}(StExp^{(*)}(A))$ is defined as the LTS generated by $\widehat{\mathscr{T}}^{(*)}(A)$, with $\downarrow \subseteq StExp^{(*)}(A)$ as set of terminating vertices, and with set $\rightarrow \subseteq StExp^{(*)}(A) \times (\underline{A} \times \mathbb{N}) \times StExp^{(*)}(A)$ of transitions, where $\underline{A} = A \cup \{1\}$.

Definition 5.12. For every star expression $e \in StExp(A)$ we denote by $\widehat{\mathcal{C}(e)}$ the entry/body-labeling that is the chart formed as the *e*-rooted sub-LTS generated by $\{e\}$ of the entry/body-labeling $\widehat{\mathcal{L}(StExp^{(*)}(A))}$.

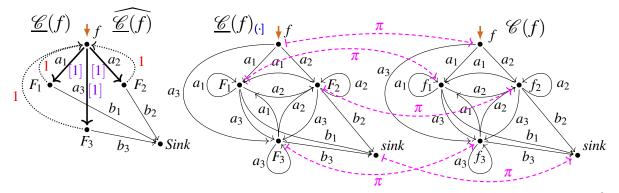
For this entry/body-labeling we will show in Section 6 that it recovers the property LEE for the stacked star expressions 1-LTS, and as a consequence, for the 1-chart interpretation of star expressions.

Lemma 5.13. $\widehat{\mathscr{L}}(StExp^{(*)}(A))$ is an entry/body-labeling, and indeed a LLEE-witness, of $\mathscr{L}(StExp^{(*)}(A))$. **Theorem 5.14.** For every $e \in StExp(A)$, the entry/body-labeling $\widehat{\mathscr{C}(e)}$ of $\mathscr{C}(e)$ is a LLEE-witness of $\mathscr{C}(e)$. Hence the 1-chart interpretation $\mathscr{C}(e)$ of a star expression $e \in StExp(A)$ satisfies the property LEE.

Example 5.15. We consider the chart interpretations $\mathscr{C}(e)$ and $\mathscr{C}(f)$ for the star expressions e and f in Ex. 2.6 for which we saw in Section 4 that LEE fails. We first illustrate the 1-chart interpretations $\underline{\mathscr{C}}(e)$ of e together with the entry/body-labeling $\underline{\mathscr{C}}(e)$ of $\underline{\mathscr{C}}(e)$. The dotted transitions indicate 1-transitions. The expressions at non-initial vertices of $\underline{\mathscr{C}}(e)$ are $E'_1 = ((1 * a^*) \cdot b^*) * e$, $E_1 = (a^* \cdot b^*) * e$, $E_2 = b^* * e$, and $E'_2 = (1 * b^*) * e$, which are obtained as iterated derivatives via the TSS $\underline{\mathscr{T}}^{(*)}$. Furthermore, we depict the induced chart $\underline{\mathscr{C}}(e)_{(\cdot)}$ of $\underline{\mathscr{C}}(e)$, and its relationship to the chart interpretation $\mathscr{C}(e)$ of e via the projection function π that defines a functional bisimulation (which we indicate via arrows $\vdash \rightarrow$).



The transitions of the induced chart $\underline{\mathscr{C}}(e)_{(\cdot]}$ of the 1-chart interpretation $\underline{\mathscr{C}}(e)$ of e correspond to paths in $\underline{\mathscr{C}}(e)$ that start with a (potentially empty) 1-transition path and have a final proper action transition, which also provides the label of the induced transition. For example the *b*-transition from E'_1 to E'_2 in $\underline{\mathscr{C}}(e)_{(\cdot]}$ arises as the induced transition in $\underline{\mathscr{C}}(e)$ that is the path that consists of the 1-transitions from E'_1 to E_1 , and from E_1 to e, followed by the final *b*-transition from e to E'_2 . The vertices with immediate termination in $\underline{\mathscr{C}}(e)_{(\cdot]}$ are all those that permit 1-transition paths in $\underline{\mathscr{C}}(e)$ to vertices with immediate termination. Therefore in $\underline{\mathscr{C}}(e)$ only e needs to permit immediate termination in order to get induced termination in $\underline{\mathscr{C}}(e)_{(\cdot]}$ also at all other vertices (like in $\mathscr{C}(e)$). Now clearly the projection function π that maps $e \mapsto e$, and $E_i, E'_i \mapsto e_i$ for $i \in \{1,2\}$ defines a bisimulation from $\underline{\mathscr{C}}(e)_{(\cdot]}$ to $\mathscr{C}(e)$. Provided that the unreachable vertices E_1 and E_2 in $\underline{\mathscr{C}}(e)_{(\cdot]}$ are removed by garbage collection, π defines an isomorphism. What is more, the entry/body-labeling $\underline{\mathscr{C}}(e)$ of $\underline{\mathscr{C}}(e)$ can be readily checked to be a LLEE-witness of $\underline{\mathscr{C}}(e)$.



Above we have illustrated the 1-chart interpretation $\underline{\mathscr{C}}(f)$ of f together with the entry/body-labeling $\underline{\mathscr{C}}(f)$ of $\underline{\mathscr{C}}(f)$, the induced chart $\underline{\mathscr{C}}(f)_{(\cdot]}$ of $\underline{\mathscr{C}}(f)$, and its connection via π to the chart interpretation $\mathscr{C}(f)$ of f: The stacked star expressions at non-initial vertices in $\underline{\mathscr{C}}(f)$ are $F_i = ((1 \cdot (1 + b_i \cdot 0)) * f_0^*) \cdot 0$ for $i \in \{1, 2, 3\}$, and $Sink = ((1 \cdot 0) * f_0^*) \cdot 0$. Here the projection function π maps $f \mapsto f$, $Sink \mapsto sink$, and $F_i \mapsto f_i$ for $i \in \{1, 2, 3\}$, defining a functional bisimulation from $\underline{\mathscr{C}}(f)_{(\cdot]}$ to $\mathscr{C}(f)$, which is a isomorphism. The entry/body-labeling $\underline{\widehat{\mathscr{C}}(f)}$ of $\underline{\mathscr{C}}(f)$ can again readily be checked to be a LLEE-witness of $\underline{\mathscr{C}}(f)$.

Thus we have verified the joint claims of Thm. 5.9 and of Thm. 5.14 for the two examples e and f of star expressions from Ex. 2.6 for which, as we saw in Section 4, LEE fails for their chart interpretations $\mathscr{C}(e)$ and $\mathscr{C}(f)$: the property LEE can be recovered for the 1-chart interpretations $\underline{\mathscr{C}}(e)$ of e, and $\underline{\mathscr{C}}(f)$ of f (by Thm. 5.14), and also, the induced charts $\underline{\mathscr{C}}(e)_{(\cdot]}$ of $\underline{\mathscr{C}}(e)$, and $\underline{\mathscr{C}}(f)_{(\cdot]}$ of $\underline{\mathscr{C}}(f)$ map to the original process interpretations $\mathscr{C}(e)$ of e, and $\mathscr{C}(f)$ of f, respectively, via a functional bisimulation (by Thm. 5.9).

6 **Proofs of the properties (P1) and (P2)**

In this section we provide the proofs of the properties (P1) and (P2) (see in the Introduction) of the variant process semantics $\underline{\mathscr{C}}(\cdot)$ in relation to the process semantics $\mathscr{C}(\cdot)$. In Section 6.1 we first sketch the crucial steps that are necessary for proving Lem. 5.8, but refer to the report version [5] for more details of the argument. From Lem. 5.8 we prove Thm. 5.9, and in this way demonstrate property (P1). In Section 6.2 we provide the details of the proofs of Lem. 5.13, and of Thm. 5.14, thereby demonstrating the property (P2).

6.1 Sketch of the proof of property (P1) of $\underline{\mathscr{C}}(\cdot)$ and $\mathscr{C}(\cdot)$

For Lem. 5.8 we need to show that the projection function $\pi : StExp^{(*)}(A) \to StExp(A)$ defines a bisimulation between the induced LTS $\underline{\mathscr{L}}(StExp^{(*)}(A))_{(\cdot)}$ of the stacked star expressions 1-LTS $\underline{\mathscr{L}}(StExp^{(*)}(A))$

and the star expressions LTS $\mathscr{L}(StExp(A))$. We show this by using statements that relate derivability in the TSS $\mathscr{T}(A)$ that defines transitions in $\mathscr{L}(StExp(A))$ to derivability in a TSS that defines transitions in the induced LTS $\mathscr{L}(StExp^{(*)}(A))_{(\cdot]}$. For this we define a TSS $\mathscr{T}^{(*)}_{(\cdot]}(A)$ as an extension of $\mathscr{T}^{(*)}(A)$ by rules that produce induced transitions and induced termination. We also formulate two easy lemmas about $\mathscr{T}^{(*)}_{(\cdot)}(A)$, where the first one states that $\mathscr{T}^{(*)}_{(\cdot)}(A)$ defines $\mathscr{L}(StExp^{(*)}(A))_{(\cdot)}$.

Definition 6.1. The LTS $\mathscr{L}_{\underline{\mathscr{T}}^{(*)}(A)} = \langle StExp^{(*)}(A), A, \downarrow^{(1)}, \xrightarrow{(\cdot)} \rangle$ is generated by derivations in the TSS $\underline{\mathscr{T}}^{(*)}(A)$ that in addition to the axioms and rules of $\underline{\mathscr{T}}^{(*)}(A)$ also contains the following rules:

$$\frac{E\downarrow}{E\downarrow^{(1)}} \quad \frac{E\xrightarrow{1}\widetilde{E}}{E\downarrow^{(1)}} \quad \frac{E\xrightarrow{a}E'}{E\downarrow^{(1)}} \quad \frac{E\xrightarrow{a}E'}{E\xrightarrow{(a]}E'} \quad \frac{E\xrightarrow{1}\widetilde{E}}{E\xrightarrow{(a]}E'} \quad \frac{E\xrightarrow{1}\widetilde{E}}{E\xrightarrow{(a]}E'}$$

Lemma 6.2. $\underline{\mathscr{L}}(StExp^{(*)}(A))_{(\cdot)} = \mathscr{L}_{\underline{\mathscr{T}}^{(*)}(A)}.$

Lemma 6.3. The following rules are admissible for $\underline{\mathscr{T}}_{(\cdot)}^{(*)}$ (they can be eliminated from $\underline{\mathscr{T}}_{(\cdot)}^{(*)}$ derivations):

$$\begin{array}{c} \underline{E_1}\downarrow^{(1)} \quad \underline{e_2}\downarrow \\ (E_1 \cdot \underline{e_2})\downarrow^{(1)} \end{array} \quad \underline{E_1}\downarrow^{(1)} \\ \hline (E_1 \ast \underline{e_2})\downarrow^{(1)} \end{array} \quad \underline{E_1} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \cdot \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \cdot \underline{e_2} \end{array} \quad \underline{E_1} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \times \underline{e_2}' \\ \hline E_1 \ast \underline{e_2} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \xrightarrow{E_1} \stackrel{(a)}{\longleftarrow} E_1' \\ \hline E_1 \xrightarrow{E_1} \stackrel{(a)}{\longrightarrow} E_1' \\ \hline E_1 \xrightarrow{E_1} \stackrel{(a)}{\longleftarrow} \underbrace{E_1} \stackrel{(a)}{\longleftarrow} \underbrace{E_1} \xrightarrow{E_1} \stackrel{(a)}{\longleftarrow} \underbrace{E_1} \xrightarrow{E_1}$$

For the proofs of these two lemmas, please see the report version [5]. Next we formulate and outline the proof of a crucial lemma that relates derivability statements in $\underline{\mathscr{T}}_{(4)}^{(*)}(A)$ with derivability statements in $\underline{\mathscr{T}}_{(4)}^{(*)}(A)$ in a way that will enable us to show that the projection function π defines a bisimulation between the LTSs generated by these two TSSs.

Lemma 6.4. For all $E, E' \in StExp^{(*)}(A)$, $e' \in StExp(A)$, and $a \in A$ the following statements hold concerning derivability in the TSS $\mathcal{T}_{(\cdot)}^{(*)}(A)$ and derivability in the TSS $\mathcal{T}(A)$ (we drop A from their designations):

 \vdash

$$\vdash_{\underline{\mathscr{I}}^{(*)}_{(!)}} E \downarrow^{(1)} \implies \vdash_{\mathscr{T}} \pi(E) \downarrow, \qquad (1)$$

$$F_{\underline{\mathscr{T}}^{(*)}_{(\cdot)}} E \xrightarrow{(a)} E' \implies \vdash_{\mathscr{T}} \pi(E) \xrightarrow{a} \pi(E'), \qquad (2)$$

$$\vdash_{\underline{\mathscr{T}}_{(\cdot)}^{(*)}} E \downarrow^{(\cdot)} \iff \vdash_{\mathscr{T}} \pi(E) \downarrow, \qquad (3)$$

$$\exists E' \in StExp^{(*)} \left[\pi(E') = e' \land \vdash_{\underline{\mathscr{T}}^{(*)}} E \xrightarrow{(a]} E' \right] \iff \vdash_{\mathscr{T}} \pi(E) \xrightarrow{a} e'.$$
(4)

Proof outline (for the full proof, see the report version [5]). For all $E, E' \in StExp^{(*)}(A)$, $e' \in StExp(A)$, and $a \in A$ the following implications hold from derivability in the TSS $\underline{\mathscr{T}}^{(*)}(A)$ to derivability in the TSS $\mathcal{T}(A)$ (again we drop A from their designations):

$$\vdash_{\mathscr{T}^{(*)}} E \downarrow \implies E \in StExp(A) \land \vdash_{\mathscr{T}} E \downarrow,$$
(5)

$$\vdash_{\underline{\mathscr{T}}^{(*)}} E \xrightarrow{a} E' \implies \vdash_E \pi(E) \xrightarrow{a} \pi(E'),$$
(6)

$$\vdash_{\underline{\mathscr{T}}^{(*)}} E \xrightarrow{1} \widetilde{E} \implies \left(\vdash_{\mathscr{T}} \pi(\widetilde{E}) \downarrow \implies \vdash_{\mathscr{T}} \pi(E) \downarrow \right), \tag{7}$$

$$\vdash_{\underline{\mathscr{T}}^{(*)}} E \xrightarrow{1} \widetilde{E} \implies \left(\vdash_{\mathscr{T}} \pi(\widetilde{E}) \xrightarrow{a} e' \implies \vdash_{\mathscr{T}} \pi(E) \xrightarrow{a} e' \right). \tag{8}$$

Statements (5) and (6) can be established by rule inspection, statements (7) and (8) can be shown by induction on the structure of E.

Then statement (1) of the lemma can be shown from (5) and (7), and statement (2) from (6) and (8). Statements (3) and (4) of the lemma can be shown by means of a proof by induction on the structure of *E* that makes crucial use of the admissible rules of $\mathcal{T}_{(:)}^{(*)}$ in Lem. 6.3.

Proof of Lem. 5.8. By transferring the four statements of Lem. 6.4 from the TSSs $\underline{\mathscr{T}}_{(:)}^{(*)}(A)$ and $\mathscr{T}(A)$ to their generated LTSs $\mathscr{L}_{\underline{\mathscr{T}}_{(:)}^{(*)}(A)}$ and $\mathscr{L}_{\mathscr{T}(A)}$, and by using $\mathscr{L}_{\underline{\mathscr{T}}_{(:)}^{(*)}(A)} = \mathscr{L}(StExp^{(*)}(A))$ from Lem. 6.2, and $\mathscr{L}_{\mathscr{T}(A)} = \mathscr{L}(StExp(A))$ from Def. 5.3 we obtain the following statements, for all stacked star expressions $E, E' \in StExp^{(*)}(A)$, star expressions $e' \in StExp(A)$, and actions $a \in A$:

$$E \xrightarrow{(a]} E' \text{ in } \mathscr{L}(\underline{\mathscr{T}}^{(*)}(A))_{\{\cdot\}} \implies \pi(E) \xrightarrow{a} \pi(E') \text{ in } \mathscr{L}(StExp(A)),$$
(9)

$$\exists E' \in StExp^{(*)} \left[\pi(E') = e' \land \\ E \xrightarrow{(a)} E' \text{ in } \mathscr{L}(StExp^{(*)}(A))_{(\cdot]} \right] \iff \pi(E) \xrightarrow{a} e' \text{ in } \mathscr{L}(StExp(A)),$$
(10)

$$E\downarrow^{(1)} \text{ in } \mathscr{L}(StExp^{(*)}(A))_{(\cdot)} \iff \pi(E)\downarrow \text{ in } \mathscr{L}(StExp(A)).$$
(11)

Hereby (9) follows from (2), (10) from (4), and (11) from (1) and (3). These three statements witness the forth, the back, and the termination condition in Def. 2.7 for the graph of the projection function π , which also is non-empty. Therefore π defines a bisimulation between $\mathscr{L}(StExp^{(*)}(A))_{(4)}$ and $\mathscr{L}(StExp(A))$.

Proof of Thm. 5.9. Let $e \in StExp(A)$ be a star expression. By Def. 5.5, the 1-chart interpretation $\mathcal{C}(e)$ of e with start vertex e is the $\{e\}$ -generated sub-1-chart of the stacked star expressions 1-LTS $\mathcal{L}(StExp^{(*)}(A))$. It follows by the definition of induced transitions in Def. 5.6 that the induced chart $\mathcal{C}(e)_{(:)}$ of $\mathcal{C}(e)$ is the $\{e\}$ -generated subchart of the induced LTS $\mathcal{L}(StExp^{(*)}(A))_{(:)}$ of $\mathcal{L}(StExp^{(*)}(A))$. On the other hand, the chart interpretation $\mathcal{C}(e)$ of e is the $\{e\}$ -generated subchart of the star expressions LTS $\mathcal{L}(StExp(A))$ by Def. 2.5. Now since, due to Lem. 5.8, the graph of the projection function π defines a bisimulation from $\mathcal{L}(StExp^{(*)}(A))_{(:)}$ to the star expressions LTS $\mathcal{L}(StExp(A))$ which due to $\pi(e) = e$ relates e with itself, it follows that the restriction of π to the $\{e\}$ -generated part $\mathcal{C}(e)_{(:)}$ of $\mathcal{L}(StExp(A))_{(:)}$ defines a bisimulation to the $\{e\}$ -generated part $\mathcal{C}(e)$ of $\mathcal{L}(StExp(A))$. From this we conclude that $\mathcal{L}(e)_{(:)} \rightarrow \mathcal{C}(e)$ holds.

6.2 Proof of property (P2) of $\underline{\mathscr{C}}(\cdot)$

We now turn to proving Lem. 5.13, which states that the entry/body-labeling $\underline{\mathscr{L}}(StExp^{(*)}(A))$ from Def. 5.11 is a LLEE-witness for the stacked star expression 1-LTS $\underline{\mathscr{L}}(StExp^{(*)}(A))$. As a preparation for verifying that the properties of a LLEE-witness are fulfilled, we prove a number of technical statements in the two lemmas below. For this, we introduce the set AppCxt(A) of *applicative contexts of stacked star expressions* over A by which we mean the set of contexts that are defined by the grammar:

 $C[\cdot] ::= \Box \mid C[\cdot] \cdot e \mid C[\cdot] * e^*$ (where $e \in StExp(A)$).

In view of Def. 2.1 every stacked star expression $E \in StExp^{(*)}(A)$ can be parsed uniquely as of the form E = C[e] for some star expression $e \in StExp(A)$, and an applicative context $C[\cdot] \in AppCxt(A)$.

Lemma 6.5. If $E \to_l E'$ in $\widehat{\mathscr{L}}(StExp^{(*)}(A))$, then also $C[E] \to_l C[E']$, for each $l \in \{bo\} \cup \{[n] \mid n \in \mathbb{N}^+\}$.

Proof. By induction on the structure of applicative contexts, using the rules for \cdot and * of $\widehat{\underline{\mathscr{T}}}^{(*)}(A)$.

Lemma 6.6. (a) Every maximal \rightarrow_{bo} path from $C[E * e^*]$ in $\widehat{\mathscr{L}}(StExp^{(*)}(A))$, where $E \in StExp^{(*)}(A)$, $e \in StExp(A)$, and $C[\cdot] \in AppCxt(A)$, is of either of the following two forms:

- (i) $C[E * e^*] = C[E_0 * e^*] \rightarrow_{bo} C[E_1 * e^*] \rightarrow_{bo} \dots \rightarrow_{bo} C[E_n * e^*] \rightarrow_{bo} \dots$ is finite or infinite with $E_0, E_1, \dots, E_n, \dots \in StExp^{(*)}(A)$ such that $E_0 \rightarrow_{bo} E_1 \rightarrow_{bo} \dots \rightarrow_{bo} E_n \rightarrow_{bo} \dots$,
- (*ii*) $C[E * e^*] = C[E_0 * e^*] \rightarrow_{\text{bo}} C[E_1 * e^*] \rightarrow_{\text{bo}} \dots \rightarrow_{\text{bo}} C[E_n * e^*] \rightarrow_{\text{bo}} C[e^*] \rightarrow_{\text{bo}} \dots$ for $n \in \mathbb{N}$, $E_0, E_1, \dots, E_n \in StExp^{(*)}(A)$ with $E_0 \rightarrow_{\text{bo}} E_1 \rightarrow_{\text{bo}} \dots \rightarrow_{\text{bo}} E_n$, $E_n \downarrow$, and $E_n * e^* \rightarrow_{\text{bo}} e^*$.

- (b) Every maximal \rightarrow_{bo} path from $E \cdot e$ in $\widehat{\mathscr{L}}(StExp^{(*)}(A))$, for $E \in StExp^{(*)}(A)$, and $e \in StExp(A)$, where now no filling in an applicative context is permitted, is of either of the following two forms:
 - (i) $E \cdot e = E_0 \cdot e \rightarrow_{bo} E_1 \cdot e \rightarrow_{bo} \dots \rightarrow_{bo} E_n \cdot e \rightarrow_{bo} \dots$ is finite or infinite with stacked star expressions $E_0, E_1, \dots, E_n, \dots \in StExp^{(*)}(A)$ such that $E_0 \rightarrow_{bo} E_1 \rightarrow_{bo} \dots \rightarrow_{bo} E_n \rightarrow_{bo} \dots$,
 - (ii) $E \cdot e = E_0 \cdot e \to_{bo} E_1 \cdot e \to_{bo} \dots \to_{bo} E_n \cdot e \to_{bo} E' \to_{bo} \dots$ with $n \in \mathbb{N}$, and stacked star expressions $E_0, E_1, \dots, E_n, F \in StExp^{(*)}(A)$ such that $E_0 \to_{bo} E_1 \to_{bo} \dots \to_{bo} E_n, E_n \downarrow$ and $e \to_l E'$.

Proof. We first argue for statement (b). Due to the rules for \cdot in $\widehat{\mathscr{T}}^{(*)}$ it holds for every step $E \cdot e \to_{bo} G$ either (1) $E \to_{bo} E_1$, and $G = E_1 \cdot e$, for some E_1 , or (2) $E \downarrow$, and $e \to_l F$, and G = F, for some F and a marking label l. In case (2) we have recognized any maximal path $E \cdot e \to_{bo} G \to_{bo} \ldots$ as of the form (ii) of item (b) for n = 0. In case (1) such a maximal path is parsed as $E \cdot e \to_{bo} E_1 \cdot e = G_1 \to_{bo} \ldots$. Then we can use the same argument again for the second step $E_1 \cdot e \to_{bo} G_2$ of that path. So by parsing a (finite or infinite) maximal path $E \cdot e \to_{bo} G_1 \to_{bo} \ldots \to_{bo} G_n \to_{bo} \ldots$ with this argument over its steps from left to right, we either discover at some finite stage that one of its steps is of case (2), and then can conclude that the path is of form (ii), or we only encounter steps of case (1) and thereby parse the path as of form (i).

We turn to statement (a). We observe that any first step $C[E * e^*] \rightarrow_{bo} G_1$ can, in view the TSS rules for \cdot and * and since $E * e^*$ does not terminate immediately, only arise from a step $E * e^* \rightarrow_{bo} F$ via context filling as $C[E * e^*] \rightarrow_{bo} C[F] = G_1$. Therefore it suffices to show the statement for (b) for the empty context $C = \Box$. For this restricted statement we can argue analogously as for item (a). Here we use that due to the TSS rules for * it holds for every step $E * e^* \rightarrow_{bo} G$ that either (1) $E \rightarrow_{bo} E_1$, and $G = E_1 * e$, for some E_1 , or (2) $E \downarrow$, and $E * e^* \rightarrow_{bo} e^*$, and $G = e^*$. By parsing the steps of a maximal \rightarrow_{bo} path $E * e^* \rightarrow_{bo} G_1 \rightarrow_{bo} \dots \rightarrow_{bo} G_n \rightarrow_{bo} \dots$ from left to right we either encounter a step of case (2), and then find that the path is of form (ii), or otherwise we find that the path is of form (i) in statement (a). \Box

Lemma 6.7. The following statements hold for paths of transitions in $\widehat{\mathscr{L}}(StExp(A))$:

- (i) There are no infinite \rightarrow_{bo} paths in $\widehat{\mathscr{L}}(StExp(A))$.
- (ii) If $E \in StExp^{(*)}(A)$ is normed, then $E \to_{ho}^{*} f$ for some $f \in StExp(A)$ with $f \downarrow$.
- (iii) If $E \to_{[n]} E'$ with n > 0 and $E, E' \in StExp^{(*)}(A)$, then $E = C[e^*]$, $e \text{ normed}^+$, $e \to_l E'_0$, $E' = C[E'_0 * e^*]$, and $n = |e|_* + 1$, for some $e \in StExp(A)$, $C[\cdot] \in AppCxt(A)$, and $e'_0 \in StExp^{(*)}(A)$.
- (iv) Neither \rightarrow_{bo} and $\rightarrow_{[n]}$ steps in $\widehat{\mathscr{L}}(StExp(A))$, where $n \ge 1$, increase the star height of expressions.

Proof. For statement (i) we prove that there is no infinite \rightarrow_{bo} path from any $F \in StExp^{(*)}(A)$, by structural induction on $F \in StExp^{(*)}(A)$. For F = 0 and F = 1 there are no steps possible at all, because there is no rule with conclusion $F \stackrel{a}{=}_{i} F'$ in the TSS $\widehat{\mathcal{T}}^{(*)}(A)$. For F = a, where $a \in A$, the single step possible is $F \stackrel{a}{\rightarrow}_{bo} 1$, which cannot be extended by any further steps. For $F = e_1 + e_2$ it holds, due to the TSS rule for +, that every \rightarrow_{bo} path $F \rightarrow_{bo} F' \rightarrow_{bo} \ldots$ from F gives rise to also a path $e_i \rightarrow_{bo} F' \rightarrow_{bo} \ldots$ from e_i , for $i \in \{1, 2\}$. Since by induction hypothesis the \rightarrow_{bo} path from e_i cannot be infinite, this follows also for the \rightarrow_{bo} path from F. Now we let $F = E \cdot e$ for $E \in StExp^{(*)}(A)$ and $e \in StExp(A)$. Then every maximal \rightarrow_{bo} path from E, or 1 plus the finite length of a \rightarrow_{bo} path from E or from e is finite, it follows that every \rightarrow_{bo} path from F must have finite length as well. For the remaining case $F = E \cdot e^*$, for $E \in StExp^{(*)}(A)$ and $e \in StExp^{(*)}(A)$, we can argue analogously by using Lem. 6.6, (a). This concludes the proof of item (i).

Item (ii) can be shown by induction on the structure of $E \in StExp^{(*)}(A)$. We only consider the nontrivial case in which $E = E_0 * e^*$, for *E* normed. We have to show that there is $f \in StExp(A)$ with $E \to_{bo}^* f$ and $f \downarrow$. We first observe that unless $E_0 \downarrow$ holds, any step from $E = E_0 * e^*$ must, due to the TSS rules, be of the form $E_0 * e^* \rightarrow_l E'_0 * e^*$ for some $E'_0 \in StExp^{(*)}(A)$ such that also $E_0 \rightarrow_l E'_0$. Therefore the assumption that E is normed implies that also E_0 is normed. Then we can apply the induction hypothesis for E_0 in order to obtain a path $E_0 \rightarrow_{b_0}^* e'_0$ and $e'_0 \downarrow$ for some $e'_0 \in StExp(A)$. From this we get $E_0 * e^* \rightarrow_{b_0}^* e'_0 * e^*$ by Lem. 6.5, and $e'_0 * e^* \rightarrow_{b_0} e^*$ due to $e'_0 \downarrow$. Consequently we find $E = E_0 * e^* \rightarrow_{b_0}^* e^*$ and $e^* \downarrow$, and hence by letting $f := e^* \in StExp(A)$ we have obtained the proof obligation $E \rightarrow_{b_0}^* f$ and $f \downarrow$ in this case.

Statement (iii) can be proved by induction on the depth of derivations in the TSS $\widehat{\mathscr{T}}^{(*)}$. This proof employs two crucial facts. First, every entry-step is created by a step $e^* \to_{[n]} E' \cdot e^*$ with $e \to_l E'$ and $n = |e^*|_* = |e|_* + 1$, for some $e \in StExp(A)$, $E' \in StExp^{(*)}(A)$, and marking label *l*. And second, entry-steps are preserved under contexts $\Box \cdot e$ and $\Box * e^*$, and hence are preserved under applicative contexts.

Statement (iv) can be proved by induction on the depth of derivations in the TSS $\underline{\widehat{\mathscr{T}}}^{(*)}$.

Proof of Lem. 5.13. That $\widehat{\mathscr{L}}(StExp^{(*)}(A))$ is an entry/body-labeling of $\mathscr{L}(StExp^{(*)}(A))$ is a consequence of the fact that the rules of the TSS $\widehat{\mathscr{T}}^{(*)}(A)$ are marking labeled versions of the rules of the TSS $\widehat{\mathscr{T}}^{(*)}(A)$.

It remains to show that $\widehat{\mathscr{L}} := \widehat{\mathscr{L}}(StExp^{(*)}(A))$ is a LLEE-witness of the stacked star expressions 1-LTS $\mathscr{L} = \mathscr{L}(StExp^{(*)}(A))$. Instead of verifying the LLEE-witness conditions (W1), (W2), (W3), we establish the following four equivalent conditions that are understood to be universally quantified over all $E, E_1, F, F_1 \in StExp^{(*)}(A)$, and $n, m \in \mathbb{N}$ with n, m > 0:

(LLEE-1) $E \rightarrow_{[n]} E_1 \implies E \rightarrow_{[n]} \cdot \rightarrow_{bo}^* E$,

(LLEE-2) \rightarrow_{bo} is terminating from *E*,

(LLEE-3) $E \xrightarrow[\mathbf{+}(E)]{\mathbf{+}(E)} \stackrel{*}{\to} F \Longrightarrow F \ddagger$ (the premise means that F is in $\mathscr{C}_{\underline{\mathscr{G}}}(E,n)$ such that $F \neq E$), (LLEE-4) $E \xrightarrow[\mathbf{+}(E)]{\mathbf{-}(E)} \stackrel{*}{\to} F \rightarrow_{[m]} F_1 \Longrightarrow n > m$,

where $\xrightarrow{\mathbf{t}(E)} [n]$ and $\xrightarrow{\mathbf{t}(E)} b_0$ mean $\rightarrow_{[n]}$ and \rightarrow_{bo} steps, respectively, that avoid *E* as their targets. Hereby

(LLEE-2) obviously implies (W1). For each entry identifier $\langle E, n \rangle \in E(\underline{\mathscr{L}})$ it is not difficult to check that (LLEE-1), (LLEE-2), and (LLEE-3) imply that $\mathscr{C}_{\underline{\mathscr{L}}}(E,n)$ satisfies the loop properties (L1), (L2), and (L3), respectively, to obtain (W2). Finally, (LLEE-4) is an easy reformulation of the condition (W3).

Now (LLEE-2) is guaranteed by Lem. 6.7, (i). It remains to verify the other three conditions above. For reasoning about transitions in $\widehat{\mathscr{L}}$ we employ easy properties that follow from the rules of $\widehat{\mathscr{T}}^{(*)}(A)$.

For showing (LLEE-1), we suppose that $E \to_{[n]} E_1$ for some $E, E_1 \in StExp^{(*)}(A)$. We have to show $E \to_{[n]} \cdot \to_{bo}^* E$. By Lem. 6.7, (iii), $E \to_{[n]} E_1$ implies that $E = C[e^*]$, e normed⁺, $n = |e|_* + 1$, for some $e \in StExp(A)$ and $C[\cdot] \in AppCxt(A)$. Since e is normed⁺, and because no 1-transitions can emanate from e, there must be some normed $G \in StExp^{(*)}(A)$, and $a \in A$ such that $e \xrightarrow{a}_{l} G$ holds. We pick G accordingly. Together with $n = |e|_* + 1$ this implies $e^* \to_{[n]} G * e^*$. Since G is normed, by Lem. 6.7, (ii) there are $k \in \mathbb{N}, G_0, \ldots, G_k \in StExp^{(*)}(A)$ such that $G = G_0 \to_{bo} \ldots \to_{bo} G_k$ and $G_k \downarrow$. Then $G_k * e^* \to_{bo} e^*$, and, by Lem. 6.5, $E = C[e^*] \to_{[n]} C[G_0 * e^*] \to_{bo} \ldots \to_{bo} C[G_k * e^*] \to_{bo} C[e^*] = E$. Hence $E \to_{[n]} \cdot \to_{bo}^* E$.

For showing (LLEE-3), we suppose that $E \to_{[n]} E_1 \to_{bo} \dots \to_{bo} E_k = F$ for $E, E_1, \dots, E_k, F \in StExp^{(*)}(A)$ with $E_1, \dots, E_k, F \neq E$, and $k \in \mathbb{N}, k \ge 1$. We have to show that $F \ddagger$. By applying Lem. 6.7, (iii), $E \to_{[n]} E_1$ implies that $E = C[e^*]$, $E_1 = C[G * e^*]$, $e^* \to_{[n]} G * e^*$, for some $e \in StExp(A)$, $G \in StExp^{(*)}(A)$, and $C[\cdot] \in AppCxt(A)$. As now the considered path is of the form $E = C[e^*] \to_{[n]} E_1 = C[G * e^*] \to_{bo} E_2 \to_{bo}$ $\dots \to_{bo} E_k = F$, it follows from Lem. 6.6, (a), due to $E_1, \dots, E_k \neq E = C[e^*]$ that this path must be of form (i) there: $E = C[e^*] \to_{[n]} E_1 = C[G_0 * e^*] \to_{bo} E_2 = C[G_1 * e^*] \to_{bo} \dots \to_{bo} E_k = C[G_k * e^*] = F$ for some $G_0, G_1, \dots, G_k \in StExp^{(*)}(A)$ with $G_0 = G$. From the form $C[G_k * e^*]$ of F we can now conclude that $F \ddagger$, since a stacked star expression that is not also a star expression does not permit immediate termination.

For showing (LLEE-4), we suppose that $E \rightarrow_{[n]} E_1 \rightarrow_{bo} \dots \rightarrow_{bo} E_k = F \rightarrow_{[m]} F_1$ for some $k, m, n \in \mathbb{N}$, $k, m, n \geq 1$, and $E, E_1, \dots, E_k, F, F_1 \in StExp^{(*)}(A)$ with $E_1, \dots, E_k \neq E$. We have to show that n > m. Arguing analogously as for (LLEE-3) with Lem. 6.7, (iii), and Lem. 6.6, (a), we find that the path is of the form $E = C[e^*] \rightarrow_{[n]} E_1 = C[G_0 * e^*] \rightarrow_{bo} E_2 = C[G_1 * e^*] \rightarrow_{bo} \dots \rightarrow_{bo} E_k = C[G_k * e^*] = F \rightarrow_{[m]} F_1$ for some $G_0, G_1, \dots, G_k \in StExp^{(*)}(A)$ such that $n = |e|_* + 1, e \rightarrow_l G_0$ and $G_0 \rightarrow_{bo} G_1 \rightarrow_{bo} \dots \rightarrow_{bo} G_{k-1} =$ $F \rightarrow_{[m]} F_1$. Due to Lem. 6.7, (iv), it follows that $n = |e|_* + 1 > |e|_* \geq |G_0|_* \geq |G_1|_* \geq \dots \geq |G_k|_*$. Since due to the loop step $C[G_k * e^*] \rightarrow_{[m]} F_1$ can only originate from G_k , we conclude by Lem. 6.7, (iii), that $|G_k|_* \geq m$. From what we have obtained before, we conclude n > m.

By having verified (LLEE-1)–(LLEE-4) we have shown that $\widehat{\mathscr{L}}(StExp(A))$ is a LLEE-witness.

Proof of Thm. 5.14. Since $\widehat{\mathscr{L}}(StExp^{(*)}(A))$ is an entry/body-labeling of $\mathscr{L}(StExp^{(*)}(A))$ by Lem. 5.13, it follows that $\widehat{\mathscr{C}(e)}$, the *e*-rooted sub-1-chart of $\widehat{\mathscr{L}}(StExp^{(*)}(A))$, is an entry/body-labeling of $\mathscr{C}(e)$, the *e*-rooted subchart of the LTS $\mathscr{L}(StExp^{(*)}(A))$. Now LLEE-witnesses are preserved under taking generated subcharts, which can be concluded directly from the alternative characterization of LLEE-witnesses (LLEE-1)–(LLEE-4) in the proof of Lem. 5.13. It follows that $\widehat{\mathscr{C}(e)}$ is a LLEE-witness of $\mathscr{L}(e)$.

7 Conclusion

As shown in [6, 7], process graphs (charts) that are defined by a TSS formulation (similar to Def. 2.3) of Milner's process interpretation for '1-free star expressions' satisfy the loop existence and elimination property LEE. But for process graph interpretations of arbitrary star expressions this is not the case in general. We provided counterexamples for this fact in Section 4. By the development in Section 5 and the proofs in Section 6, however, we showed that LEE can still by made applicable for star expressions from the full class: we defined a variant $\underline{\mathscr{C}}(\cdot)$ of the process semantics $\mathscr{C}(\cdot)$ such that $\underline{\mathscr{C}}(\cdot)$ guarantees LEE.

The solution of the axiomatization problem (A) for 1-free star expressions in [6, 7] is based on transferring 'provable solutions' between bisimilar charts in order to show that they are provably equal. A provable solution is a function from vertices of a chart to star expressions that provably satisfies, in each vertex v of the chart, a correctness condition that requires that the solution value at v is provably equal to the sum of expressions $a_i \cdot w_i$ over all transitions $\langle v, a_i, w_i \rangle$ from v, plus 1 if $v \downarrow$ holds. We used six lemmas: (I) the chart interpretation $\mathscr{C}(e)$ of a 1-free star expression e is a LLEE-chart; (SI) every 1-free star expression e is the start value of a provable solution of its chart interpretation $\mathscr{C}(e)$; (C) the bisimulation collapse of a LLEE-chart is again a LLEE-chart; (E) from every LLEE-chart \mathscr{C} a provable solution of \mathscr{C} can be extracted; (P) all provable solutions can be pulled back from the target to the source chart of a functional bisimulation to obtain a provable solution of the source chart; (SE) all provable solutions of a LLEE-chart are provably equal. From these parts it follows that if $\mathscr{C}(e_1) \simeq \mathscr{C}(e_2)$ holds for 1-free star expressions e_1 and e_2 , then a proof of $e_1 = e_2$ in the axiomatization can be constructed from a solution of the bisimulation collapse \mathscr{C}_0 of $\mathscr{C}(e_1)$ and $\mathscr{C}(e_2)$, where \mathscr{C}_0 is a LLEE-chart due to (I) and (C). A solution of the recognizability problem (R) for charts concerning expressibility by 1-free star expressions can be obtained by using (C) and showing that LEE can be tested for charts in polynomial time.

Concerning the extension of this approach to the set of all star expressions we can report the following. The lemmas (E), (P), and (SE) can be extended (with technical efforts) to versions (E)₁, (P)₁, and (SE)₁ that apply to 1-charts and LLEE-1-charts. While (I) does not hold for all star expressions (see Section 4), we obtained as adaptation in Section 5: (I)₁ the 1-chart interpretation $\underline{\mathscr{C}}(e)$ of a star expression *e* is a LLEE-1-chart. An analogue version (SI)₁ of (SI) can also be shown. The remaining obstacle for extending the approach to all star expressions is that a generalization of (C) to LLEE-1-charts is not straightforward.

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