Towards Automated Readable Proofs of Ruler and Compass Constructions

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Although there are several systems that successfully generate construction steps for ruler and compass construction problems, none of them provides readable synthetic correctness proofs for generated constructions. In the present work, we demonstrate how our triangle construction solver ArgoTriCS can cooperate with automated theorem provers for first order logic and coherent logic so that it generates construction correctness proofs, that are both human-readable and formal (can be checked by interactive theorem provers such as Coq or Isabelle/HOL). These proofs currently rely on many high-level lemmas and our goal is to have them all formally shown from the basic axioms of geometry.

1 Introduction

Geometry construction problems are usually solved in four phases:

1. *Analysis*: In this phase, the geometric figure to be constructed is analyzed. The specific constraints that apply to this figure and the relationships between its elements are identified. By understanding the requirements and constraints, the steps required to construct the desired figure can be determined.
2. *Construction*: Once the problem is analyzed, the sequence of ruler and compass construction steps used to construct the figure is identified.
3. *Proof*: After the figure is constructed, it should be proved that it satisfies the properties and conditions given by the specification. Proofs in ruler and compass constructions often involve using geometric principles, such as the properties of angles, congruence, or similarity. A formal proof can be used to demonstrate the validity of the construction and ensure that it meets the desired criteria.
4. *Discussion*: The discussion phase involves reflection on the construction, its properties, and the relevant insights. It is often discussed under which condition does the solution exist and whether it is unique. Non-degeneracy conditions are also identified.

In our previous work we have described our system ArgoTriCS that can perform triangle constructions both in Euclidean geometry [6] and in absolute and hyperbolic geometry [8]. Problems from the Wernick's list [10] are analyzed and in Euclidean setting ArgoTriCS manages to solve 66 out of 74 non-isomorphic problems. Essentially it performs the problem analysis based on its internal set of definitions

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and lemmas, and finds a series of construction steps required to construct a triangle with a given set of significant points (e.g., vertices, orthocenter, centroid, centers of inscribed and circumscribed circles etc.). However it did not generate classic, readable, synthetic construction proofs. In her PhD thesis [7], Marinković describes how theorem provers, based on algebraic methods such as Wu's method [11] and Gröbner basis method [1], and semi-synthetic methods such as area method [4], integrated within GLCL tool [2] and OpenGeoProver [5], could be employed to check the construction correctness. The problem with this approach is that generated proofs are not human-readable. Since the main usage scenario of automated construction solver is in education, it is vital that students understand why some construction is correct. Therefore, obtaining human-readable proofs is of a great importance.

In the current work, we describe how an automated system such as ArgoTriCS can be combined with first-order logic and coherent logic provers so that each generated construction is accompanied by its human-readable proof of correctness. This is a work in progress, and we will describe our approach, prototype implementation, and preliminary results for a small set of selected problems.

2 Examples

Example 2.1. Consider constructing a triangle ABC given its vertex A, altitude foot H_a and circumcenter O. ArgoTriCS finds the following construction, illustrated in Figure 1:

1. Construct the line $l_1 = AH_a$.
2. Construct the line l_2 such that it is perpendicular to the line l_1 and that it contains H_a.
3. Construct the circle c centered at O containing A.
4. Let B and C be the intersections of the line l_2 and the circle c.

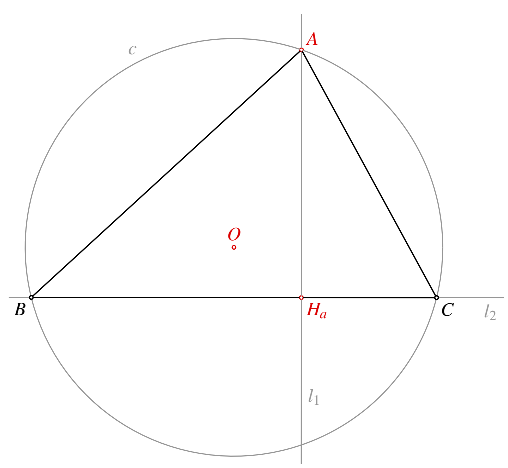

Fig. 1: Construction of the triangle ABC given the points A, O, and H_a.

Proof. We need to show that A is the vertex of the constructed triangle ABC (which is trivial), that H_a is its altitude foot and that O is its circumcenter. This proof is rather straightforward.

By construction, the circle c contains all three vertices A, B, and C, so it must be the circumcircle of the triangle ABC (since the circumcircle of a triangle is unique). The O is the center of c, so it must be the circumcenter (since the center of any circle is unique).

By construction the line l_2 contains the vertices B and C, so it must be equal to the side a of the triangle ABC (since the triangle side through the points B and C is unique). By construction the line l_1 contains A and is perpendicular to $l_2 = a$, so it must be equal to the altitude h_a (since there is a unique altitude from the vertex A). Since by construction H_a belongs both to $l_2 = a$ and $l_1 = h_a$ it must be the altitude foot H_a (since it is the unique intersection of a and h_a). □

If we analyze the previous proof, we see that it essentially relies on several uniqueness lemmas and that it merely reverses the chain of deduction steps used in the analysis phase.

In some cases, however, the proof is very different from the analysis.

Example 2.2. Consider constructing a triangle ABC given its vertex A, circumcenter O and centroid G. The construction that ArgoTriCS finds is the following (see Figure 2):

1. Construct the point P_1 such that $\overrightarrow{AG} : \overrightarrow{AP_1} = 2 : 3$.
2. Construct the point P_2 such that $\overrightarrow{OG} : \overrightarrow{OP_2} = 1 : 3$.
3. Construct the line $l_1 = AP_2$.
4. Construct the line l_2 such that it is perpendicular to the line l_1 and that it contains P_1.
5. Construct the circle c centered at O containing A.
6. Let B and C be the intersections of the line l_2 with the circle c.

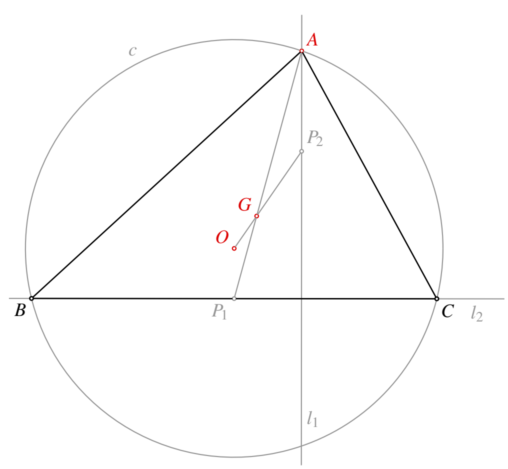

Fig. 2: Construction of the triangle ABC given the points A, O, and G.

Please note that there is a simpler solution to this construction problem, but we wanted to discuss this solution because the proof here is quite different from the construction.

Proof. We need to prove that A is the vertex of the triangle ABC (which is trivial), that G is its centroid and that O is its circumcenter. The latter is very simple (similar to the previous proof), since by construction all points A, B, and C lie on the circle c centered at O.

The line l_2 is equal to the triangle side a, since it contains the vertices B and C (and the triangle side through the points B and C is unique). By construction l_1 contains A and is perpendicular to $l_2 = a$, so it must be equal to the altitude h_a (since the altitude from vertex A is unique).

Consider line $l_3 = OP_1$. We shall prove that it is parallel to the line $l_1 = h_a$. Since by construction it holds that $\overrightarrow{OG} : \overrightarrow{OP_2} = 1 : 3$, by the elementary properties of vector ratio it also holds that $\overrightarrow{OG} : \overrightarrow{GP_2} = 1 : 2$. Similarly, it holds that $\overrightarrow{P_1G} : \overrightarrow{GA} = 1 : 2$. The angles OGP_1 and OGP_2 are opposite and therefore congruent. Hence triangles OGP_1 and P_2GA are similar, and angles OP_1G and GAP_2 are always equal, so the lines $OP_1 = l_3$ and $AP_2 = l_1 = h_a$ are parallel.

Since h_a is perpendicular to $l_2 = a$, so must be $l_3 = OP_1$. Therefore, the line l_3 must be the perpendicular bisector of the segment BC (since it is the unique line containing circumcenter O that is perpendicular to a). Consequently, the point P_1 must be equal to M_a – the midpoint of BC (as it is the unique intersection of the segment with its pependicular bisector). Finally, the point G must be the centroid of ABC since the centroid is the unique point for which it holds that $\overrightarrow{AG} : \overrightarrow{AM_a} = 2 : 3$. □

3 Automation

Our main goal is to obtain proofs such as the previous ones automatically, using coherent logic provers.

3.1 Problem Statement and Lemmas

The first step would be to make ArgoTriCS generate the problem statement, along with the construction steps. For example, the problem statement for the first problem can be given as follows:

$$\begin{aligned}
&\mathrm{inc}(A,l_1) \wedge \mathrm{inc}(H'_a,l_1) \wedge \\
&\mathrm{perp}(l_2,l_1) \wedge \mathrm{inc}(H'_a,l_2) \wedge \\
&\mathrm{circle}(O',A,c) \wedge \\
&\mathrm{inc}(B,l_2) \wedge \mathrm{inc}(C,l_2) \wedge \mathrm{inc_c}(B,c) \wedge \mathrm{inc_c}(C,c) \wedge B \neq C \Longrightarrow \\
&H'_a = H_a \wedge O' = O
\end{aligned}$$

The predicate $\mathrm{inc}(P,l)$ denotes that the point P is incident to the line l i.e., $P \in l$, $\mathrm{inc\_c}(P,c)$ denotes that the point P is incident to the circle c i.e., $P \in c$, $\mathrm{circle}(O,P,c)$ denotes that c is the circle centered at the point O passing through the point P, and $\mathrm{perp}(l_1,l_2)$ denotes that lines l_1 and l_2 are perpendicular. The point O is the real circumcenter of the triangle ABC (this is implicitly given by the lemmas that are given to the prover along with the problem statement), and H_a is the real altitude foot. For simplicity various non-degeneracy conditions are added to the problem statement (e.g., the conditions $H'_a \neq A$, $A \neq B$, $A \neq C$, etc.) before it is given to the automated theorem prover.

Along with the problem statement, automated prover is given a series of carefully chosen lemmas, that are treated as axioms. Most of those lemmas follow from the general geometric knowledge, but are instantiated for the significant points, lines and circles of the triangle ABC. Each significant object is denoted by a constant (e.g., bc for the side BC, O for the circumcenter, M_a for the midpoint of BC, h_a for the altitude from A, H_a for its foot, c° for the circumcircle etc.). Lemmas that encode properties of those objects are used both in analysis (by the ArgoTriCS) and in proofs (by automated theorem provers). Some of those lemmas are:

$$\mathrm{inc}(B,bc) \quad \wedge \quad \mathrm{inc}(C,bc)$$

$$\begin{array}{rcl} \mathrm{inc}(A,h_a) & \wedge & \mathrm{perp}(h_a,bc) \\ \overrightarrow{AG}:\overrightarrow{AM_a} & = & 2:3 \\ \mathrm{inc\_c}(A,c^\circ) & \wedge & \mathrm{inc\_c}(B,c^\circ)\wedge \mathrm{inc\_c}(C,c^\circ) \end{array}$$

However, proofs require additional lemmas that guarantee uniqueness of objects. For example:

$$\begin{array}{rcl} (\forall l)(\mathrm{inc}(A,l)\wedge \mathrm{perp}(l,bc) & \Longrightarrow & l=h_a) \\ (\forall c)(\mathrm{inc\_c}(A,c)\wedge \mathrm{inc\_c}(B,c)\wedge \mathrm{inc\_c}(C,c) & \Longrightarrow & c=c^\circ) \end{array}$$

Notice that uniqueness lemmas are given in instantiated way, meaning that they hold for some specific objects. This choice was made in order to follow the implementation of ArgoTriCS, where most of the knowledge is given in an instantiated way. However, the uniqueness axioms could be given also in more general way.

Some general lemmas about properties of basic geometric predicates are also needed. For example:

$$\begin{array}{rcl} (\forall l_1,l_2)(\mathrm{perp}(l_1,l_2) & \Longrightarrow & \mathrm{perp}(l_2,l_1)) \\ (\forall P_1,P_2)(\exists l)(\mathrm{inc}(P_1,l) & \wedge & \mathrm{inc}(P_2,l)) \end{array}$$

All those lemmas are formulated as axioms and the problem statement is formulated as a conjecture in TPTP format.[1] That file is then given to some automated theorem prover. In our experiments we used Vampire [9] and Larus [3]. Vampire is a very efficient, award winning FOL theorem prover. Its main drawback is that it cannot generate readable proofs. We also used Larus [3] that is a coherent-logic prover able to generate readable proofs and also formal proofs that can be checked by interactive theorem provers such as Isabelle/HOL or Coq.

Our second example uses the notion of ratio of vectors. However neither Vampire nor Larus have a native support for arithmetic calculations. Therefore we introduced separate predicates for ratios that frequently occur in geometric constructions (e.g., 1 : 2, 1 : 3, 2 : 3) and added lemmas that connect those ratios. For example:

$$(\forall A,B,C)(\mathrm{ratio13}(A,B,A,C)\Longrightarrow \mathrm{ratio12}(A,B,B,C))$$

The proof uses a result that follows from triangle similarity. We encoded this in the following lemma:

$$\begin{array}{l} (\forall A,M,B,X,Y,ax,by) \\ (\mathrm{ratio21}(A,M,M,B)\wedge \mathrm{ratio21}(X,M,M,Y)\wedge \\ \mathrm{line}(A,X,ax)\wedge \mathrm{line}(B,Y,by)\Longrightarrow \mathrm{para}(ax,by)) \end{array}$$

Also, in Euclidean geometry there are clear connections between parallel and perpendicular lines.

$$(\forall l1,l2,a)\ (\mathrm{perp}(l_1,a)\wedge \mathrm{para}(l_1,l_2)\Longrightarrow \mathrm{perp}(l_2,a))$$

3.2 Using Automated Provers

The conjecture of the construction problem considered in Example 2.1 can be formulated in TPTP format in the following way:

[1] `https://www.tptp.org/`

```
fof(th_A_Ha_O, conjecture, ( ( inc(pA,ha1) & inc(pHa1,ha1)
    & perp(ha1,a1) & inc(pHa1,a1) & inc_c(pA,cc1) & center(pOc1,cc1)
    & inc_c(pB,cc1) & inc(pB,a1) & inc_c(pC,cc1) & inc(pC,a1) )
    => ( pHa = pHa1 &  pOc = pOc1 ) ) ).
```

where pHa and pOc are defined by the axioms as the foot of the altitude from vertex A to side BC and circumcenter of triangle ABC, respectively.

Larus successfully proved given conjecture as two separate statements, one for each of the facts in the conclusion. Key fragment of generated readable proof is given below (all used geometry axioms are listed, others are the ones implied by equality):

Axioms:

1. bc_unique : $\forall L\ (inc(pB,L) \wedge inc(pC,L) \Rightarrow L = bc\ )$
2. haA : $\forall H\ (perp(H,bc) \wedge inc(pA,H) \Rightarrow ha = H\ )$
3. pHa_def : $\forall H1\ (inc(H1,ha) \wedge inc(H1,bc) \Rightarrow H1 = pHa\ )$
4. cc_unique : $\forall C\ (inc\_c(pA,C) \wedge inc\_c(pB,C) \wedge inc\_c(pC,C) \Rightarrow C = cc\ )$
5. center_unique : $\forall C\ \forall C1\ \forall C2\ (center(C1,C) \wedge center(C2,C) \Rightarrow C1 = C2\ )$

Example 3.1. th_A_Ha_O0 :
$inc(pA,ha1) \wedge inc(pHa1,ha1) \wedge perp(ha1,a1) \wedge inc(pHa1,a1) \wedge inc\_c(pA,cc1)$
$\wedge\ center(pOc1,cc1) \wedge inc\_c(pB,cc1) \wedge inc(pB,a1) \wedge inc\_c(pC,cc1) \wedge inc(pC,a1)$
$\Rightarrow pHa = pHa1$

Proof:

1. $pHa = pHa$ (by MP, using axiom eqnativeEqSub0; instantiation: $A \mapsto pHa$, $B \mapsto pHa$, $X \mapsto pHa$)
2. $a1 = bc$ (by MP, from $inc(pB,a1)$, $inc(pC,a1)$ using axiom bc_unique; instantiation: $L \mapsto a1$)
3. $perp(ha1,bc)$ (by MP, from $perp(ha1,a1)$, $a1 = bc$ using axiom perpEqSub1; instantiation: $A \mapsto ha1$, $B \mapsto a1$, $X \mapsto bc$)
4. $ha = ha1$ (by MP, from $perp(ha1,bc)$, $inc(pA,ha1)$ using axiom haA; instantiation: $H \mapsto ha1$)
5. $inc(pHa1,ha)$ (by MP, from $inc(pHa1,ha1)$, $ha = ha1$ using axiom incEqSub1; instantiation: $A \mapsto pHa1$, $B \mapsto ha1$, $X \mapsto ha$)
6. $inc(pHa1,bc)$ (by MP, from $inc(pHa1,a1)$, $a1 = bc$ using axiom incEqSub1; instantiation: $A \mapsto pHa1$, $B \mapsto a1$, $X \mapsto bc$)
7. $pHa1 = pHa$ (by MP, from $inc(pHa1,ha)$, $inc(pHa1,bc)$ using axiom pHa_def; instantiation: $H1 \mapsto pHa1$)
8. $pHa = pHa1$ (by MP, from $pHa1 = pHa$, $pHa = pHa$ using axiom eqnativeEqSub0; instantiation: $A \mapsto pHa$, $B \mapsto pHa1$, $X \mapsto pHa$)
9. Proved by assumption! (by QEDas)

Example 3.2. th_A_Ha_O1 :
$inc(pA,ha1) \wedge inc(pHa1,ha1) \wedge perp(ha1,a1) \wedge inc(pHa1,a1) \wedge inc\_c(pA,cc1)$
$\wedge\ center(pOc1,cc1) \wedge inc\_c(pB,cc1) \wedge inc(pB,a1) \wedge inc\_c(pC,cc1) \wedge inc(pC,a1)$
$\Rightarrow pOc = pOc1$

Proof:

1. $center(pOc,cc)$ (by MP, using axiom centerEqSub1; instantiation: $A \mapsto pOc$, $B \mapsto cc$, $X \mapsto cc$)
2. $cc1 = cc$ (by MP, from $inc\_c(pA,cc1)$, $inc\_c(pB,cc1)$, $inc\_c(pC,cc1)$ using axiom cc_unique; instantiation: $C \mapsto cc1$)

3. $center(pOc1, cc)$ (by MP, from $center(pOc1, cc1)$, $cc1 = cc$ using axiom centerEqSub1; instantiation: $A \mapsto pOc1$, $B \mapsto cc1$, $X \mapsto cc$)

4. $pOc = pOc1$ (by MP, from $center(pOc, cc)$, $center(pOc1, cc)$ using axiom center_unique; instantiation: $C \mapsto cc$, $C1 \mapsto pOc$, $C2 \mapsto pOc1$)

5. Proved by assumption! (by QEDas)

Correctness proof of the generated construction for the problem considered in Example 2.2 is given in Appendix.

4 Results

We considered the subset of problems from Wernick's corpus, over vertices of the triangle, midpoints of triangle sides, feet of altitudes, centroid, circumcenter and orthocenter of the triangle. It consists of 35 non-isomorphic location triangle problems. For each of these problems, we tried to prove the correctness of constructions found by ArgoTriCS using FOL prover Vampire and coherent logic prover Larus. Vampire succesfully proved 31 of these problems, while Larus proved 20 problems, and for remaining ones it could not prove it in given timelimit.

5 Conclusion

Although this is a work-in-progress, we have managed to show that this approach is plausible and can be used to automatically obtain readable proofs of correctness for geometric constructions. This is very important in the context of mathematical education, where students need to know why a geometric statement holds. In our previous work, we have described ArgoTriCS – a system that is able to perform ruler and compass construction steps for almost all solvable problems in the Wernick's corpus [6, 10]. The main step in the ArgoTriCS implementation was to formulate a good set of lemmas to be used for analysing and finding the construction. This work shows that an identified set of lemmas is not sufficient to generate correctness proofs, and that the proof phase requires an additional set of lemmas (mainly the lemmas that guarantee uniqueness, but also some other equally important lemmas). However, once these lemmas are identified, they can be passed to general-purpose theorem provers, which can then generate fully synthetic proofs of correctness. Although the coherent logic solvers we have tested are not yet as powerful as the FOL solvers such as Vampire, if they succeed in solving the given problem, they provide us with human-readable proofs.

A very important issue is the correctness of the used lemmas. Indeed, if some lemmas are incorrect (e.g., if a precondition or a non-degeneracy condition is missing), a contradiction may arise and the theorem could be proved from this contradiction. We examined all the generated proofs, and all of them were correct. To be completely sure that our lemmas are correct, we formalize them in Isabelle/HOL and prove them using the axioms of geometry. Since Larus can output Isabelle/HOL proofs, we will eventually have a system capable of generating proofs of construction that are fully mechanically verified starting from the axioms.

In the present work we have not considered degenerate cases and the existence of constructed objects (we have simply assumed that everything is non-degenerate and that all constructed objects exist). However, we plan to pay more attention to this issue and extend our tools to perform the final discussion phase where they would automatically identify the necessary non-degeneracy conditions.

Coherent logic prover, Larus, used in this research is currently unable to find all correctness proofs fully automatically. We have worked around this by giving it hints in the form of lemmas. We plan to use other coherent logic provers, and we are in contact with the Larus developers so that they can improve their prover using the feedback they have received from our problems.

A Appendix

Larus cannot currently prove the whole theorem only if no guidance is provided. Therefore, we first derive several lemmas and then use those lemmas to prove the main theorem. The first part of the conjecture is easily proved:

Axioms:

1. cc_unique : $\forall C\ (inc\_c(pA,C) \wedge inc\_c(pB,C) \wedge inc\_c(pC,C) \Rightarrow C = cc\ )$
2. center_unique : $\forall C\ \forall C1\ \forall C2\ (center(C1,C) \wedge center(C2,C) \Rightarrow C1 = C2\ )$
3. bc_unique : $\forall L\ (inc(pB,L) \wedge inc(pC,L) \Rightarrow L = bc\ )$
4. haA : $\forall H\ (perp(H,bc) \wedge inc(pA,H) \Rightarrow ha = H\ )$
5. inc_line : $\forall P1\ \forall P2\ \forall L\ (inc(P1,L) \wedge inc(P2,L) \wedge P1 \neq P2 \Rightarrow line(P1,P2,L)\ )$
6. ex_line : $\forall P1\ \forall P2\ (\exists L\ (line(P1,P2,L))\ )$
7. ratio21_para : $\forall A\ \forall G\ \forall Ma\ \forall H\ \forall Oc\ \forall Lba\ \forall Lha\ (ratio21(A,G,G,Ma) \wedge ratio21(H,G,G,Oc) \wedge line(Oc,Ma,Lba) \wedge line(A,H,Lha) \Rightarrow para(Lba,Lha)\ )$
8. perp_para : $\forall Lba\ \forall Lha\ \forall A\ (perp(Lha,A) \wedge para(Lba,Lha) \Rightarrow perp(Lba,A)\ )$
9. perp_unique : $\forall P\ \forall L\ \forall L1\ \forall L2\ (perp(L1,L) \wedge inc(P,L1) \wedge perp(L2,L) \wedge inc(P,L2) \Rightarrow L1 = L2\ )$
10. pMa_is_interect_bisa_bc : $\forall P\ (inc(P,bc) \wedge inc(P,bisa) \Rightarrow P = pMa\ )$

Example A.1. th_A_O_G_1:

$ratio23(pA,pG1,pA,pMa1) \wedge ratio23(pH1,pG1,pH1,pOc1) \wedge inc(pA,ha1) \wedge inc(pH1,ha1) \wedge inc(pMa1,a1) \wedge perp(a1,ha1) \wedge center(pOc1,cc1) \wedge inc\_c(pA,cc1) \wedge inc\_c(pB,cc1) \wedge inc(pB,a1) \wedge inc\_c(pC,cc1) \wedge inc(pC,a1) \wedge pA \neq pH1 \Longrightarrow pOc1 = pOc$

Proof:

1. $cc1 = cc$ (by MP, from $inc\_c(pA,cc1)$, $inc\_c(pB,cc1)$, $inc\_c(pC,cc1)$ using axiom cc_unique; instantiation: $C \mapsto cc1$)
2. $center(pOc1,cc)$ (by MP, from $center(pOc1,cc1)$, $cc1 = cc$ using axiom centerEqSub1; instantiation: $A \mapsto pOc1$, $B \mapsto cc1$, $X \mapsto cc$)
3. $pOc1 = pOc$ (by MP, from $center(pOc1,cc)$ using axiom center_unique; instantiation: $C \mapsto cc$, $C1 \mapsto pOc1$, $C2 \mapsto pOc$)
4. Proved by assumption! (by QEDas)

Then, the facts `a1 = bc` and `ha1 = ha` can be derived:

Example A.1. *lm_A_O_G_2:*

$ratio23(pA,pG1,pA,pMa1) \wedge ratio23(pH1,pG1,pH1,pOc1) \wedge inc(pA,ha1) \wedge inc(pH1,ha1) \wedge inc(pMa1,a1) \wedge perp(a1,ha1) \wedge center(pOc1,cc1) \wedge inc\_c(pA,cc1) \wedge inc\_c(pB,cc1) \wedge inc(pB,a1) \wedge inc\_c(pC,cc1) \wedge inc(pC,a1) \wedge pA \neq pH1 \Longrightarrow a1 = bc$

Proof:

1. $a1 = bc$ (by MP, from $inc(pB,a1)$, $inc(pC,a1)$ using axiom bc_unique; instantiation: $L \mapsto a1$)
2. Proved by assumption! (by QEDas)

Example A.2. *lm_A_O_G_3:*
$ratio23(pA, pG1, pA, pMa1) \wedge ratio23(pH1, pG1, pH1, pOc1) \wedge inc(pA, ha1) \wedge inc(pH1, ha1) \wedge inc(pMa1, a1) \wedge perp(a1, ha1) \wedge center(pOc1, cc1) \wedge inc\_c(pA, cc1) \wedge inc\_c(pB, cc1) \wedge inc(pB, a1) \wedge inc\_c(pC, cc1) \wedge inc(pC, a1) \wedge pA \neq pH1 \Longrightarrow ha1 = ha$

Proof:

1. $a1 = bc$ (by MP, from $inc(pB,a1)$, $inc(pC,a1)$ using axiom bc_unique; instantiation: $L \mapsto a1$)
2. $perp(bc, ha1)$ (by MP, from $perp(a1,ha1)$, $a1 = bc$ using axiom perpEqSub0; instantiation: $A \mapsto a1$, $B \mapsto ha1$, $X \mapsto bc$)
3. $ha = ha1$ (by MP, from $perp(bc,ha1)$, $inc(pA,ha1)$ using axiom haA; instantiation: $H \mapsto ha1$)
4. $ha1 = ha$ (by MP, from $ha = ha1$ using axiom eq_sym; instantiation: $A \mapsto ha$, $B \mapsto ha1$)
5. Proved by assumption! (by QEDas)

Now the conclusions of these lemmas can be added to the set of premises, and the next lemma can be proved:

Example A.3. *lm_A_O_G_4 :*
$ratio23(pA, pG1, pA, pMa1) \wedge ratio23(pH1, pG1, pH1, pOc1) \wedge inc(pA, ha1) \wedge inc(pH1, ha1) \wedge inc(pMa1, a1) \wedge perp(a1, ha1) \wedge center(pOc1, cc1) \wedge inc\_c(pA, cc1) \wedge inc\_c(pB, cc1) \wedge inc(pB, a1) \wedge inc\_c(pC, cc1) \wedge inc(pC, a1) \wedge pA \neq pH1 \wedge pOc1 = pOc \wedge a1 = bc \wedge ha1 = ha \Longrightarrow line(pOc1, pMa1, bisa)$

Proof:

1. $inc(pOc1, bisa)$ (by MP, from $pOc1 = pOc$ using axiom incEqSub0; instantiation: $A \mapsto pOc$, $B \mapsto bisa$, $X \mapsto pOc1$)
2. Let w be such that $line(pOc1, pMa1, w)$ (by MP, using axiom ex_line; instantiation: $P1 \mapsto pOc1$, $P2 \mapsto pMa1$)
3. $line(pA, pH1, ha1)$ (by MP, from $inc(pA,ha1)$, $inc(pH1,ha1)$, $pA \neq pH1$ using axiom inc_line; instantiation: $P1 \mapsto pA$, $P2 \mapsto pH1$, $L \mapsto ha1$)
4. $para(w, ha1)$ (by MP, from $ratio23(pA,pG1,pA,pMa1)$, $ratio23(pH1,pG1,pH1,pOc1)$, $line(pOc1,pMa1,w)$, $line(pA,pH1,ha1)$ using axiom ratio21_para; instantiation: $A \mapsto pA$, $G \mapsto pG1$, $Ma \mapsto pMa1$, $H \mapsto pH1$, $Oc \mapsto pOc1$, $Lba \mapsto w$, $Lha \mapsto ha1$)
5. $perp(ha1, bc)$ (by MP, from $ha1 = ha$ using axiom perpEqSub0; instantiation: $A \mapsto ha$, $B \mapsto bc$, $X \mapsto ha1$)
6. $perp(w, bc)$ (by MP, from $perp(ha1,bc)$, $para(w,ha1)$ using axiom perp_para; instantiation: $Lba \mapsto w$, $Lha \mapsto ha1$, $A \mapsto bc$)
7. $w = bisa$ (by MP, from $perp(w,bc)$, $line(pOc1,pMa1,w)$, $inc(pOc1,bisa)$ using axiom perp_unique; instantiation: $P \mapsto pOc1$, $L \mapsto bc$, $L1 \mapsto w$, $L2 \mapsto bisa$)
8. $line(pOc1, pMa1, bisa)$ (by MP, from $line(pOc1,pMa1,w)$, $w = bisa$ using axiom lineEqSub2; instantiation: $A \mapsto pOc1$, $B \mapsto pMa1$, $C \mapsto w$, $X \mapsto bisa$)
9. Proved by assumption! (by QEDas)

Finally, with the conclusion of this lemma added to the premises, we can prove the final statament:

Example A.2. th_A_O_G_5:

$ratio23(pA, pG1, pA, pMa1) \land ratio23(pH1, pG1, pH1, pOc1) \land inc(pA, ha1) \land inc(pH1, ha1) \land inc(pMa1, a1) \land perp(a1, ha1) \land center(pOc1, cc1) \land inc\_c(pA, cc1) \land inc\_c(pB, cc1) \land inc(pB, a1) \land inc\_c(pC, cc1) \land inc(pC, a1) \land pA \neq pH1 \land pOc1 = pOc \land a1 = bc \land ha1 = ha \land pOc = pOc1 \land line(pOc1, pMa1, bisa) \implies pG = pG1$

Proof:

1. $inc(pMa1, bc)$ (by MP, from $inc(pMa1, a1)$, $a1 = bc$ using axiom incEqSub1; instantiation: $A \mapsto pMa1$, $B \mapsto a1$, $X \mapsto bc$)
2. $pMa1 = pMa$ (by MP, from $inc(pMa1, bc)$, $line(pOc1, pMa1, bisa)$ using axiom pMa_is_interect_bisa_bc; instantiation: $P \mapsto pMa1$)
3. $ratio23(pA, pG1, pA, pMa)$ (by MP, from $ratio23(pA, pG1, pA, pMa1)$, $pMa1 = pMa$ using axiom ratio23EqSub3; instantiation: $A \mapsto pA$, $B \mapsto pG1$, $C \mapsto pA$, $D \mapsto pMa1$, $X \mapsto pMa$)
4. $pG = pG1$ (by MP, from $ratio23(pA, pG1, pA, pMa)$ using axiom ratio23_Ma_Gsat0; instantiation: $X \mapsto pG1$)
5. Proved by assumption! (by QEDas)

References

[1] Bruno Buchberger (2006): *An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal*. Journal of Symbolic Computation 41(3), pp. 475–511, doi:10.1016/j.jsc.2005.09.007.

[2] Predrag Janičić (2006): *GCLC – A Tool for Constructive Euclidean Geometry and More than That*. In: Proceedings of International Congress of Mathematical Software (ICMS 2006), Lecture Notes in Computer Science 4151, pp. 58–73, doi:10.1007/11832225_6.

[3] Predrag Janičić & Julien Narboux (2021): *Automated Generation of Illustrations for Synthetic Geometry Proofs*. In: Proceedings of the 13th International Conference on Automated Deduction in Geometry, ADG 2021, EPTCS 352, pp. 91–102, doi:10.4204/EPTCS.352.9.

[4] Predrag Janičić, Julien Narboux & Pedro Quaresma (2012): *The Area Method - a recapitulation*. Journal of Automated Reasoning 48(4), pp. 489–532, doi:10.1007/s10817-010-9209-7.

[5] Filip Marić, Ivan Petrović, Danijela Petrović & Predrag Janičić (2012): *Formalization and Implementation of Algebraic Methods in Geometry*. In: Proceedings First Workshop on CTP Components for Educational Software, Electronic Proceedings in Theoretical Computer Science 79, pp. 63–81, doi:10.4204/EPTCS.79.4.

[6] Vesna Marinković (2017): *ArgoTriCS – Automated Triangle Construction Solver*. Journal of Experimental & Theoretical Artificial Intelligence 29(2), pp. 247–271, doi:10.1080/0952813X.2015.1132271.

[7] Vesna Marinković (2015): *Automated Solving of Construction Problems in Geometry*. Ph.D. thesis, University of Belgrade.

[8] Vesna Marinković, Tijana Šukilović & Filip Marić (2023): *Automated Triangle Constructions in Hyperbolic Geometry*. to appear in Annals of Mathematics and Artificial Intelligence, doi:10.1007/s10472-023-09850-5.

[9] Alexandre Riazanov & Andrei Voronkov (2002): *The design and implementation of Vampire*. AI Communications 15(2,3), pp. 91–110.

[10] William Wernick (1982): *Triangle Constructions with Three Located Points*. Mathematics Magazine 55(4), pp. 227–230, doi:10.1080/0025570X.1985.11976988.

[11] Wen Tsun Wu (1978): *On the Decision Problem and the Mechnization of Theorem-proving in Elementary Geometry*. Scientia Sinica 21(2), pp. 159–172.