

Input-Driven Double-Head Pushdown Automata

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We introduce and study input-driven deterministic and nondeterministic double-head pushdown automata. A double-head pushdown automaton is a slight generalization of an ordinary pushdown automaton working with two input heads that move in opposite directions on the common input tape. In every step one head is moved and the automaton decides on acceptance if the heads meet. Demanding the automaton to work input-driven it is required that every input symbol uniquely defines the action on the pushdown store (push, pop, state change). Normally this is modeled by a partition of the input alphabet and is called a *signature*. Since our automaton model works with two heads either both heads respect the same signature or each head owes its own signature. This results in two variants of input-driven double-head pushdown automata. The induced language families on input-driven double-head pushdown automata are studied from the perspectives of their language describing capability, their closure properties, and decision problems.

1 Introduction

Input-driven pushdown automata were introduced in [11] in the course of deterministic context-free language recognition by using a pebbling strategy on the mountain range of the pushdown store. The idea on input driven pushdown automata is that the input letters uniquely determine whether the automaton pushes a symbol, pops a symbol, or leaves the pushdown unchanged. The follow-up papers [3] and [6] studied further properties of the family of input-driven pushdown languages. One of the most important properties on input-driven pushdown languages is that deterministic and nondeterministic automata are equally powerful. Moreover, the language family accepted is closed under almost all basic operations in formal language theory. Although the family of input-driven pushdown languages is a strict subset of the family of deterministic context-free languages, the input-driven pushdown languages are still powerful enough to describe important context-free-like structures and moreover share many desirable properties with the family of regular languages. These features turned out to be useful in the context of program analysis and led to a renewed interest [1] on input-driven pushdown languages about ten years ago. In [1] an alternative name for input-driven pushdown automata and languages was coined, namely visibly pushdown automata and languages. Sometimes input-driven pushdown languages are also called nested word languages. Generally speaking, the revived research on input-driven pushdown languages triggered the study of further input-driven automata types, such as input-driven variants of, e.g., (ordered) multi-stack automata [5], stack automata [2], queue automata [9], etc.

We contribute to this list of input-driven devices, by introducing and studying input-driven double-head pushdown automata. Double-head pushdown automata were recently introduced in [13].¹ Instead of reading the input from left to right as usual, in a double-head pushdown automata the input is processed from the opposite ends of the input by double-heads, and the automaton decides on acceptance

¹Originally these devices were named two-head pushdown automata in [13], but since this naming may cause confusion with multi-head pushdown automata of [7], we use to refer to them as double-head pushdown automata instead.

when the two heads meet. Thus, double-head pushdown automata are a straight forward generalization of Rosenberg’s double-head finite automata for linear context-free languages [15]—see also [12]. The family of double-head nondeterministic pushdown languages is a strict superset of the family of context-free languages and contains some linguistically important non-context-free languages. In fact, the family of double-head nondeterministic pushdown languages forms a mildly context-sensitive language family because in addition to the aforementioned containment of important languages, the word problem of double-head nondeterministic pushdown languages remains solvable in deterministic polynomial time as for ordinary pushdown automata. Moreover, every double-head nondeterministic pushdown language is semi-linear. Double-head pushdown automata are a moderate extension of ordinary pushdown automata because languages accepted by double-head pushdown automata still satisfy an iteration or pumping lemma. Thus, double-head pushdown automata and the properties of their accepted languages are interesting objects to study.

In the next section we introduce the necessary notations on double-head pushdown automata and their input-driven versions. Demanding the automaton to work input-driven it is required that every input symbol uniquely defines the action on the pushdown store (push, pop, state change). Normally this is modeled by a partition of the input alphabet and is called a *signature*. Since our automaton model works with two heads either both heads respect the same signature or each head owes its own signature. This results in (simple) input-driven and double input-driven double-head pushdown automata. Then in Section 3 we investigate the computational capacity of (double) input-driven double-head pushdown automata. We show that nondeterministic machines are more powerful than deterministic ones, for both input-driven variants. Moreover, it turns out that the language families in question are incomparable to classical language families such as the growing context-sensitive languages, the Church-Rosser languages, and the context-free languages. As a byproduct we also separate the original language families of double-head deterministic and double-head nondeterministic pushdown languages. Section 4 is then devoted to the closure properties of the families of input driven double-head pushdown languages and finally in Section 5 we consider decision problems for the language families in question. Here it is worth mentioning that although some problems are already not semidecidable even for deterministic machines, the question of whether a given deterministic input-driven double-head pushdown automaton M is equivalent to a given regular language is decidable. In contrast, the decidability of this question gets lost, if M is a nondeterministic input-driven double-head pushdown machine. We have to leave open the status of some decision problems such as equivalence and regularity. This is subject to further research.

2 Preliminaries

Let Σ^* denote the set of all words over the finite alphabet Σ . The *empty word* is denoted by λ , and $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$. For convenience, throughout the paper we use Σ_λ for $\Sigma \cup \{\lambda\}$. The set of words of length at most $n \geq 0$ is denoted by $\Sigma^{\leq n}$. The *reversal* of a word w is denoted by w^R . For the *length* of w we write $|w|$. For the number of occurrences of a symbol a in w we use the notation $|w|_a$. *Set inclusion* is denoted by \subseteq and *strict set inclusion* by \subset . We write 2^S for the power set and $|S|$ for the cardinality of a set S .

A *double-head pushdown automaton* is a pushdown automaton that is equipped with two read-only input heads that move in opposite directions on a common input tape. In every step one head is moved. The automaton halts when the heads would pass each other.

A pushdown automaton is called *input-driven* if the input symbols currently read define the next action on the pushdown store, that is, pushing a symbol onto the pushdown store, popping a symbol from

the pushdown store, or changing the state without modifying the pushdown store. To this end, we assume the input alphabet Σ joined with λ to be partitioned into the sets Σ_N , Σ_D , and Σ_R , that control the actions state change only (N), push (D), and pop (R).

Formally, a *nondeterministic input-driven double-head pushdown automaton* (ndet-ID2hPDA) is a system $M = \langle Q, \Sigma, \Gamma, q_0, F, \perp, \delta_D, \delta_R, \delta_N \rangle$, where Q is the finite set of *states*, Σ is the finite set of *input symbols* partitioned into the sets Σ_D , Σ_R , and Σ_N , Γ is the finite set of *pushdown symbols*, $q_0 \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *accepting states*, $\perp \notin \Gamma$ is the *empty pushdown symbol*, δ_D is the partial transition function mapping from $Q \times (\Sigma_D \cup \{\lambda\})^2 \times (\Gamma \cup \{\perp\})$ to $2^{Q \times \Gamma}$, δ_R is the partial transition function mapping from $Q \times (\Sigma_R \cup \{\lambda\})^2 \times (\Gamma \cup \{\perp\})$ to 2^Q , δ_N is the partial transition function mapping from $Q \times (\Sigma_N \cup \{\lambda\})^2 \times (\Gamma \cup \{\perp\})$ to 2^Q , where all transition functions are defined only if the second or third argument is λ , and none of the transition functions is defined for $Q \times \{\lambda\}^2 \times (\Gamma \cup \{\perp\})$.

A *configuration* of an ndet-ID2hPDA $M = \langle Q, \Sigma, \Gamma, q_0, F, \perp, \delta_D, \delta_R, \delta_N \rangle$ is a triple (q, w, s) , where $q \in Q$ is the current state, $w \in \Sigma^*$ is the unread part of the input, and $s \in \Gamma^*$ denotes the current pushdown content, where the leftmost symbol is at the top of the pushdown store. The *initial configuration* for an input string w is set to (q_0, w, λ) . During the course of its computation, M runs through a sequence of configurations. One step from a configuration to its successor configuration is denoted by \vdash . Let $a \in \Sigma$, $w \in \Sigma^*$, $z' \in \Gamma$, $s \in \Gamma^*$, and $z = \perp$ if $s = \lambda$ and $z = z_1$ if $s = z_1 s_1 \in \Gamma^+$. We set

1. $(q, aw, s) \vdash (q', w, z's)$, if $a \in \Sigma_D$ and $(q', z') \in \delta_D(q, a, \lambda, z)$,
2. $(q, wa, s) \vdash (q', w, z's)$, if $a \in \Sigma_D$ and $(q', z') \in \delta_D(q, \lambda, a, z)$,
3. $(q, aw, s) \vdash (q', w, s')$, if $a \in \Sigma_R$ and $q' \in \delta_R(q, a, \lambda, z)$,
where $s' = \lambda$ if $s = \lambda$ and $s' = s_1$ if $s = z_1 s_1 \in \Gamma^+$.
4. $(q, wa, s) \vdash (q', w, s')$, if $a \in \Sigma_R$ and $q' \in \delta_R(q, \lambda, a, z)$,
where $s' = \lambda$ if $s = \lambda$ and $s' = s_1$ if $s = z_1 s_1 \in \Gamma^+$.
5. $(q, aw, s) \vdash (q', w, s)$, if $a \in \Sigma_N$ and $q' \in \delta_N(q, a, \lambda, z)$,
6. $(q, wa, s) \vdash (q', w, s)$, if $a \in \Sigma_N$ and $q' \in \delta_N(q, \lambda, a, z)$,

So, whenever the pushdown store is empty, the successor configuration is computed by the transition functions with the special empty pushdown symbol \perp , and at most one head is moved. As usual, we define the reflexive and transitive closure of \vdash by \vdash^* . The language accepted by the ndet-ID2hPDA M is the set $L(M)$ of words for which there exists some computation beginning in the initial configuration and halting in a configuration in which the whole input is read and an accepting state is entered. Formally:

$$L(M) = \{w \in \Sigma^* \mid (q_0, w, \lambda) \vdash^* (q, \lambda, s) \text{ with } q \in F, s \in \Gamma^*\}.$$

The partition of an input alphabet into the sets Σ_D , Σ_R , and Σ_N is called a *signature*. We also consider input-driven double-head pushdown automata, where each of the two heads may have its own signature. To this end, we provide the signatures $\Sigma_{D,l}$, $\Sigma_{R,l}$, and $\Sigma_{N,l}$ as well as $\Sigma_{D,r}$, $\Sigma_{R,r}$, and $\Sigma_{N,r}$ and require for *double input-driven double-head pushdown automata* (double-ID2hPDA) that they obey the first signature whenever the left head is moved and the second signature whenever the right head is moved.

If there is at most one choice of action for any possible configuration, we call the given (double) input-driven double-head pushdown automaton *deterministic* (det-ID2hPDA or det-double-ID2hPDA).

In general, the family of all languages accepted by an automaton of some type X will be denoted by $\mathcal{L}(X)$.

In order to clarify this notion we continue with an example.

Example 1 The Gladkij language $\{w\#w^R\#w \mid w \in \{a,b\}^*\}$ is not growing context sensitive [4] and, thus, is neither context free nor Church-Rosser [10]. The same is true for its marked variant

$$L_1 = \{w\#h_1(w)^R\#h_2(w) \mid w \in \{a,b\}^*\},$$

where the homomorphisms h_1 and h_2 are defined by $h_1(a) = \bar{a}$, $h_1(b) = \bar{b}$, $h_2(a) = \hat{a}$, and $h_2(b) = \hat{b}$. However, language L_1 is accepted by the det-ID2hPDA

$$M = \langle \{q_0, q_1, q_a, q_b, q_+\}, \Sigma_D \cup \Sigma_R \cup \Sigma_N, \{A, B\}, q_0, \{q_+\}, \perp, \delta_D, \delta_R, \delta_N \rangle,$$

where $\Sigma_D = \{a, b\}$, $\Sigma_R = \{\bar{a}, \bar{b}\}$, $\Sigma_N = \{\#, \hat{a}, \hat{b}\}$, and the transition functions are defined as follows. Let $X \in \{A, B, \perp\}$.

$$\begin{array}{ll} (1) \quad \delta_D(q_0, a, \lambda, X) = (q_0, A) & (5) \quad \delta_N(q_0, \#, \lambda, X) = q_1 \\ (2) \quad \delta_D(q_0, b, \lambda, X) = (q_0, B) & (6) \quad \delta_N(q_a, \lambda, \hat{a}, X) = q_1 \\ (3) \quad \delta_R(q_1, \bar{a}, \lambda, A) = q_a & (7) \quad \delta_N(q_b, \lambda, \hat{b}, X) = q_1 \\ (4) \quad \delta_R(q_1, \bar{b}, \lambda, B) = q_b & (8) \quad \delta_N(q_1, \#, \lambda, \perp) = q_+ \end{array}$$

The idea of the construction is as follows. In a first phase, M reads and pushes the input prefix w (Transitions 1 and 2). On reading the left symbol $\#$ automaton M enters state q_1 which is used in the second phase. Basically, in the second phase the left and right head are moved alternately. When the left head is moved, the input symbol read is compared with the symbol on the top of the pushdown store. If both coincide, state q_a or q_b is entered to indicate that the right head has to read symbol \hat{a} or \hat{b} (Transitions 3 and 4). If the right head finds the correct symbol, state q_1 is entered again (Transitions 6 and 7). The second phase ends when the left head reads the second symbol $\#$. In that case state q_+ is entered and M halts (Transition 8). If in this situation the input has been read entirely and the pushdown store is empty, clearly, the w pushed in the first phase has successfully be compared with the factor $h_1(w)^R$ and the suffix $h_2(w)$. So, the input belongs to L_1 . In any other case, M halts without entering the sole accepting state q_+ . ■

3 Computational capacity

In order to explore the computational capacity of input-driven double-head pushdown automata we first turn to show that nondeterminism is better than determinism. As witness language for that result we use the language $L_{dta} = \{a^n b^n c^n \$l c^i b^j a^k \mid i, j, k, n \geq 0\} \cup \{a^i b^j c^k \$r c^n b^n a^n \mid i, j, k, n \geq 0\}$.

Lemma 2 *The language L_{dta} is not accepted by any deterministic double-head pushdown automaton.*

Proof In contrast to the assertion assume that L_{dta} is accepted by some deterministic double-head pushdown automaton M . For all $m, n \geq 0$ and $x \in \{l, r\}$ we consider the input words $a^m b^m c^m \$x c^n b^n a^n$ that belong to L_{dta} . Since M is deterministic, the computations on the words $a^m b^m c^m \$l c^n b^n a^n$ and $a^m b^m c^m \$r c^n b^n a^n$ are identical until one of the heads reaches the center marker $\$x$. So, we can define the set

$$R = \{(m, n) \mid \text{on input } a^m b^m c^m \$x c^n b^n a^n \text{ the right head of } M \text{ reaches } \$x \text{ not after the left head}\}.$$

Thus, the initial part of such a computation is in the form $(q_0, a^m b^m c^m \$x c^n b^n a^n, \lambda) \vdash^* (q_1, u \$x, z) \vdash (q_2, u, z')$, where $q_1, q_2 \in Q$, $z, z' \in \Gamma^*$, u is a suffix of $a^m b^m c^m$, and the last transition applied is of the

form $\delta(q_1, \lambda, \$x, z_1)$. That is, in the last step the right head of M reads $\$x$ while seeing z_1 on top of the pushdown store, and the pushdown store content z is replaced by z' . Next, the set R is further refined into $R(m) = \{n \mid (m, n) \in R\}$, for all $m \geq 0$. Clearly, we have $R = \bigcup_{m \geq 0} (m, R(m))$.

Now assume that there is an $m \geq 0$ such that $|R(m)|$ is infinite. We sketch the construction of a deterministic pushdown automaton M_1 that accepts the language $L_1 = \{a^j b^j c^j \$r \mid j \in R(m)\}$ as follows. On a given input $a^j b^j c^j \$r$, M_1 basically simulates a computation of M on input $a^m b^m c^m \$r c^j b^j a^j$. Since m is fixed, M_1 handles the prefix $a^m b^m c^m$ in its finite control. Moreover, since the left head of M reaches the center marker not before the right head, M_1 handles the left head and its moves in the finite control as well. So, whenever M moves its right head to the left, M_1 moves its sole head to the right. Let $[u, q]$ denote a state of M_1 that says that q is the simulated state of M and u is the still unprocessed suffix of the prefix $a^m b^m c^m$. Then the simulation of M is straightforward: If M performs a computation $(q_0, a^m b^m c^m \$r c^j b^j a^j, \lambda) \vdash (q_1, u \$r v, z)$, where u is a suffix of $a^m b^m c^m$ and v is a prefix of $c^j b^j a^j$, then M_1 performs a computation $([a^m b^m c^m, q_0], c^j b^j a^j \$r, \lambda) \vdash ([u, q_1], (\$r v)^R, z)$. When M_1 has read the symbol $\$r$, it continues the simulation of M with λ -steps, where now all head movements are handled in the finite control. Finally, M_1 accepts if and only if M accepts. So, since M_1 is a deterministic pushdown automaton, L_1 must be a context-free language. However, since $|R(m)|$ is assumed to be infinite, language L_1 is infinite. A simple application of the pumping lemma for context-free languages shows that any infinite subset of $\{a^k b^k c^k \$r \mid k \geq 0\}$ is not context free. From the contradiction we derive that $|R(m)|$ is finite, for all $m \geq 0$.

In particular, this means that for every $m \geq 0$ there is at least one $n \geq 0$ such that M accepts the input $a^m b^m c^m \$l c^n b^n a^n$, whereby the right head reaches the center marker not before the left head. Based on this fact, we now can construct a nondeterministic pushdown automaton M_2 that accepts the language $L_2 = \{a^j b^j c^j \$l \mid j \geq 0\}$ as follows.

On a given input $a^j b^j c^j \$l$, M_2 simulates a computation of M on input $a^j b^j c^j \$l c^k b^k a^m$, whereby M_2 guesses the suffix $c^k b^k a^m$ step-by-step. So, whenever M moves its left head to the right, M_2 moves its sole head to the right as well. Whenever M moves its right head to the left, M_2 guesses the next symbol from the suffix. If M_2 guesses the center marker and, thus, in the simulation the right head would see the center marker before the left head, M_2 rejects. If in the simulation the left head sees the center marker before the right head, the simulation continues with λ -steps until M_2 guesses that both heads meet. In this case, M_2 accepts if and only if M accepts. Considering the language accepted by M_2 , one sees that on some input $a^j b^j c^j \$l$, $j \geq 0$, M_2 can guess a suffix $c^n b^n a^n$ such that M accepts the input $a^j b^j c^j \$l c^n b^n a^n$, whereby the right head reaches the center marker not before the left head (it follows from above that such a suffix exists). So, M_2 accepts any word of the form $a^j b^j c^j \$l$, $j \geq 0$. Conversely, if an input $w \$l$ is not of the form $a^j b^j c^j \$l$, then there is no computation of M_2 that accepts any word with prefix $w \$l$ (due to the center marker $\$l$, w is verified to have form $a^j b^j c^j$). Therefore, the simulation cannot ending accepting and, thus, M_2 rejects.

So, since M_2 is a pushdown automaton, L_2 must be a context-free language. From the contradiction we derive that L_{dta} is not accepted by any deterministic double-head pushdown automaton. \square

The next example shows that the language L_{dta} is accepted even by input-driven double-head pushdown automata provided that nondeterminism is allowed.

Example 3 The language $L_{dta} = \{a^i b^j c^k \$r c^n b^n a^n \mid i, j, k, n \geq 0\} \cup \{a^n b^n c^n \$l c^i b^j a^k \mid i, j, k, n \geq 0\}$ is accepted by the ndet-ID2hPDA

$$M = \langle \{s_0, p_1, p_2, \dots, p_7, q_1, q_2, \dots, q_7, s_+\}, \Sigma_D \cup \Sigma_R \cup \Sigma_N, \{D, G, A\}, s_0, \{p_4, q_4, s_+\}, \perp, \delta_D, \delta_R, \delta_N \rangle,$$

where $\Sigma_D = \{a, \$l, \$r\}$, $\Sigma_R = \{b\}$, $\Sigma_N = \{c\}$, and the transition functions are defined as follows. In its first step, M guesses whether the input contains a symbol $\$r$ or a symbol $\$l$. Dependent on the guess one of the two subsets in the definition of L_{dta} are verified, and M starts to read the prefix from the left or the suffix from the right. The states p_ℓ are used in the case of a $\$r$, the states q_ℓ are used in the case of a $\$l$, and the remaining states are used for both cases. Recall that i, j, k and n may be zero.

$$\begin{array}{ll}
(1) & \delta_D(s_0, a, \lambda, \perp) \ni (p_1, D) & (5) & \delta_D(s_0, \lambda, a, \perp) \ni (q_1, D) \\
(2) & \delta_R(s_0, b, \lambda, \perp) \ni p_2 & (6) & \delta_R(s_0, \lambda, b, \perp) \ni q_2 \\
(3) & \delta_N(s_0, c, \lambda, \perp) \ni p_3 & (7) & \delta_N(s_0, \lambda, c, \perp) \ni q_3 \\
(4) & \delta_D(s_0, \$r, \lambda, \perp) \ni (p_4, G) & (8) & \delta_D(s_0, \lambda, \$l, \perp) \ni (q_4, G)
\end{array}$$

We continue to construct the transition functions for the first case, the construction for the second case is symmetric. So, while M processes the prefix $a^i b^j c^k$ it has to verify the form of the prefix and has to obey the action associated to the symbols. The actual pushdown content generated in this phase does not matter. Therefore, M pushes dummy symbols D and a special symbol G when it has reached the $\$r$. The form of the prefix is verified with the help of the states p_1, p_2 , and p_3 :

$$\begin{array}{ll}
(9) & \delta_D(p_1, a, \lambda, D) \ni (p_1, D) & (13) & \delta_R(p_2, b, \lambda, D) \ni p_2 \\
(10) & \delta_R(p_1, b, \lambda, D) \ni p_2 & (14) & \delta_N(p_2, c, \lambda, D) \ni p_3 \\
(11) & \delta_N(p_1, c, \lambda, D) \ni p_3 & (15) & \delta_D(p_2, \$r, \lambda, D) \ni (p_4, G) \\
(12) & \delta_D(p_1, \$r, \lambda, D) \ni (p_4, G) & (16) & \delta_N(p_3, c, \lambda, D) \ni p_3 \\
& & (17) & \delta_D(p_3, \$r, \lambda, D) \ni (p_4, G)
\end{array}$$

In the next phase, M has to verify the suffix $c^n b^n a^n$. To this end, it moves its both heads alternately where for each but the first a an A is pushed. For the first a (if it exists) a symbol G is pushed. So, if both heads, one after the other, arrive at the b -sequence, the number n of a 's coincides with the number of c 's and the pushdown content is of the form $A^{n-1}GGD^*$, if $n \geq 1$. If $n = 0$, M halts in the accepting state p_4 . If $n \geq 1$, in the final phase the b -sequence is read and its length is compared with the number of A 's at the top of the stack. The final step reads the last b and the G from the pushdown store and enters the accepting state s_+ , for which no further transitions are defined. So, a correct input is accepted and an incorrect input is not:

$$\begin{array}{ll}
(18) & \delta_D(p_4, \lambda, a, G) \ni (p_5, G) & (20) & \delta_R(p_6, \lambda, b, A) \ni p_7 \\
(19) & \delta_N(p_5, c, \lambda, A) \ni p_6 & (21) & \delta_R(p_6, \lambda, b, G) \ni s_+ \\
(20) & \delta_N(p_5, c, \lambda, G) \ni p_6 & (22) & \delta_R(p_7, \lambda, b, A) \ni p_7 \\
(20) & \delta_D(p_6, \lambda, a, A) \ni (p_5, A) & (23) & \delta_R(p_7, \lambda, b, G) \ni s_+ \\
(21) & \delta_D(p_6, \lambda, a, G) \ni (p_5, A)
\end{array}$$

■

By Lemma 2 and Example 3 we conclude the next theorem.

Theorem 4 *The family $\mathcal{L}(\text{det-ID2hPDA})$ is strictly included in the family $\mathcal{L}(\text{ndet-ID2hPDA})$ and the family $\mathcal{L}(\text{det-double-ID2hPDA})$ is strictly included in the family $\mathcal{L}(\text{ndet-double-ID2hPDA})$.*

Next, we compare the computational capacities of ID2hPDAs and double-ID2hPDAs, that is, the capacity gained in providing (possibly different) signatures to each of the heads. As witness language for the result that two signatures are better than one we use the language $L_{ta} = \{a^n b^n a^n \mid n \geq 1\}$.

Example 5 The language $L_{ta} = \{a^n b^n a^n \mid n \geq 1\}$ is accepted by the det-double-ID2hPDA

$$M = \langle \{q_0, q_1, q_2, q_a, q_+\}, \Sigma_{D,l} \cup \Sigma_{D,r} \cup \Sigma_{R,l} \cup \Sigma_{R,r} \cup \Sigma_{N,l} \cup \Sigma_{N,r}, \{A, G\}, q_0, \{q_+\}, \perp, \delta_D, \delta_R, \delta_N \rangle,$$

where $\Sigma_{D,l} = \{a\}$, $\Sigma_{D,r} = \emptyset$, $\Sigma_{R,l} = \{b\}$, $\Sigma_{R,r} = \{b\}$, $\Sigma_{N,l} = \emptyset$, $\Sigma_{N,r} = \{a\}$, and the transition functions are defined as follows. In its first step, M reads an a with its left head and pushes the special symbol G into the pushdown store. Subsequently, it reads an a with its right head while the pushdown store remains unchanged. Next, M moves its both heads alternately where for each a read by the left head (in state q_1) an A is pushed and for each a read by the right head (in state q_a) the pushdown store remains unchanged. Let $X \in \{A, G\}$.

$$\begin{aligned} (1) \quad \delta_D(q_0, a, \lambda, \perp) &= (q_a, G) \\ (2) \quad \delta_D(q_1, a, \lambda, X) &= (q_a, A) \\ (3) \quad \delta_N(q_a, \lambda, a, X) &= q_1 \end{aligned}$$

So, after having read $n \geq 1$ symbols a with the left as well as with the right head, M is in state q_1 and the pushdown store contains the word $A^{n-1}G$. The next phase starts when the right head reads a b in state q_1 . In this phase, the right head is not used. The left head reads the b 's while for each b an A is popped. When M reads a b with the special symbol G on top of the pushdown store, the sole accepting state q_+ is entered.

$$\begin{aligned} (4) \quad \delta_R(q_1, b, \lambda, A) &= q_2 \\ (5) \quad \delta_R(q_1, b, \lambda, G) &= q_+ \\ (6) \quad \delta_R(q_2, b, \lambda, A) &= q_2 \\ (7) \quad \delta_R(q_2, b, \lambda, G) &= q_+ \end{aligned}$$

In this way, clearly, any word from L_{ta} is accepted by M . Conversely, in order to reach its sole accepting state q_+ , M has to read at least one a from the left as well as from the right, otherwise it cannot enter state q_1 . Moreover, whenever it reads an a from the left it must read an a from the right, otherwise it would halt in state q_a . Since the transition functions are undefined for input symbol b and states q_0 and q_a , M cannot accept without having read a b . In order to enter the accepting state q_+ it must have read as many b 's as a 's from the prefix which, in turn, have been read from the suffix as well. Since the transition functions are undefined for state q_+ , M necessarily halts when this state is entered. If in this situation the input has been read entirely, M accepts. This implies that any word accepted by M belongs to L_{ta} . ■

Lemma 6 *The language L_{ta} is not accepted by any nondeterministic input-driven double-head pushdown automaton.*

By Example 5 and Lemma 6 we conclude the next theorem.

Theorem 7 *The family $\mathcal{L}(\text{det-ID2hPDA})$ is strictly included in the family $\mathcal{L}(\text{det-double-ID2hPDA})$ and the family $\mathcal{L}(\text{ndet-ID2hPDA})$ is strictly included in the family $\mathcal{L}(\text{ndet-double-ID2hPDA})$.*

So far, we have shown that nondeterminism is better than determinism for a single as well as for double signatures, and that double signatures are better than a single signature for deterministic as well as nondeterministic computations. Moreover, Lemma 6 shows that language L_{ta} is not accepted by any nondeterministic input-driven double-head pushdown automaton, while Example 5 shows that language L_{ta} is accepted, even by a deterministic input-driven double-head pushdown automaton, provided that double signatures are available. Conversely, Example 3 reveals that language L_{dta} is accepted even with a

nondeterministic input-driven double-head pushdown automaton with a single signature, while Lemma 2 shows that, to this end, determinism is not sufficient. This implies the next corollary.

Corollary 8 *The families $\mathcal{L}(\text{det-double-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are incomparable.*

Next we turn to compare the four language families under consideration with some other well-known language families.

A context-sensitive grammar is said to be *growing context sensitive* if the right-hand side of every production is *strictly* longer than the left-hand side. The family of growing context-sensitive languages (GCSL) lies strictly in between the context-free and context-sensitive languages. Another language family lying properly in between the regular and the growing context-sensitive languages are the *Church-Rosser languages* (CRL), which have been introduced in [10]. They are defined via finite, confluent, and length-reducing Thue systems. Church-Rosser languages are incomparable to the context-free languages [4] and have neat properties. For example, they parse rapidly in linear time, contain non-semilinear as well as inherently ambiguous languages [10], are characterized by deterministic automata models [4, 14], and contain the deterministic context-free languages (DCFL) as well as their reversals (DCFL^R) properly [10].

Theorem 9 *Each of the families $\mathcal{L}(\text{det-ID2hPDA})$, $\mathcal{L}(\text{ndet-ID2hPDA})$, $\mathcal{L}(\text{det-double-ID2hPDA})$, and $\mathcal{L}(\text{ndet-double-ID2hPDA})$ is incomparable with GCSL as well as with CRL.*

Proof Example 1 shows that the marked Gladkij language $L_1 = \{w\#h_1(w)^R\#h_2(w) \mid w \in \{a, b\}^*\}$, where the homomorphisms h_1 and h_2 are defined by $h_1(a) = \bar{a}$, $h_1(b) = \bar{b}$, $h_2(a) = \hat{a}$, and $h_2(b) = \hat{b}$, is accepted by some det-ID2hPDA. Language L_1 is not growing context sensitive and, thus, is not a Church-Rosser language [4].

Conversely, the unary language $\{a^{2^n} \mid n \geq 0\}$ is not semilinear, but a Church-Rosser language [10]. Since every language accepted even by some nondeterministic double-head pushdown is semilinear [13] the incomparabilities claimed follow. □

The inclusion structure of the families in question is depicted in Figure 1.

4 Closure properties

We investigate the closure properties of the language families induced by nondeterministic and deterministic input-driven 2hPDAs. Table 1 summarizes our results.

	—	∪	∩	∪REG	∩REG	·	*	<i>h</i>	<i>h</i> ⁻¹	<i>R</i>
$\mathcal{L}(2\text{hPDA})$	no	yes	no	yes	yes	no	no	yes	?	yes
$\mathcal{L}(\text{det-ID2hPDA})$	yes	no	no	yes	yes	no	no	no	no	yes
$\mathcal{L}(\text{ndet-ID2hPDA})$	no	no	no	yes	yes	no	no	no	no	yes

Table 1: Closure properties of the language classes in question.

In [13] it was shown that the family of languages accepted by ordinary nondeterministic double-head pushdown automata is closed under union, homomorphism, and reversal, but it is *not* closed under intersection, complement, concatenation, and iteration. Furthermore it was shown that the languages

$$L_1 = \{a^n b^n c^n d^n e^n \mid n \geq 1\}, \quad L_2^2, \quad \text{and} \quad L_2^*, \quad \text{for} \quad L_2 = \{a^n b^n c^n d^n \mid n \geq 1\}$$

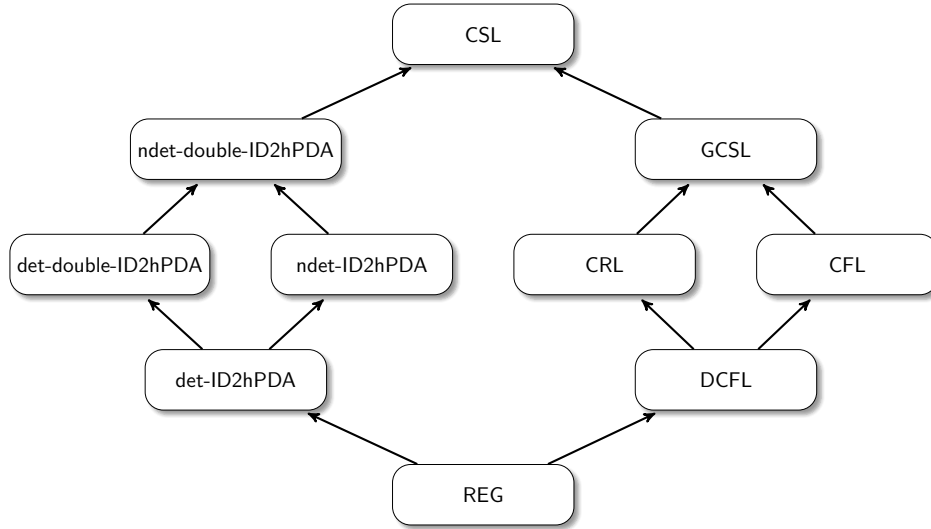


Figure 1: Inclusion structure of language families. The arrows indicate strict inclusions. All nodes not connected by a directed path are incomparable, where the incomparability with the (deterministic) context-free languages is a conjecture.

cannot be accepted by any nondeterministic double-head pushdown automaton. We start our investigation with the language families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$.

Our first result is an easy observation that if the input heads of a (input-driven) 2hPDA change their roles, the accepted language is the reversal of the original language. Hence we show closure under reversal of the language families in question.

Theorem 10 *Both language families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are closed under reversal.*

Proof Let $M = \langle Q, \Sigma, \Gamma, q_0, F, \perp, \delta_D, \delta_R, \delta_N \rangle$ be an ndet-ID2hPDA. We construct an ndet-ID2hPDA $M' = \langle Q, \Sigma, \Gamma, q_0, F, \perp, \delta'_D, \delta'_R, \delta'_N \rangle$, where the transition function is defined as follows: for every $q, q' \in Q$, $z, z' \in \Gamma^*$, and $x, y \in \Sigma$ we set

- $(q', z') \in \delta'_D(q, y, x, z)$, if $(q', z') \in \delta_D(q, x, y, z)$,
- $q' \in \delta'_R(q, y, x, z)$, if $q' \in \delta_R(q, x, y, z)$, and
- $q' \in \delta'_N(q, y, x, z)$, if $q' \in \delta_N(q, x, y, z)$.

Then it is easy to see by induction on the length of the computation that M' accepts the reversal of the language $L(M)$, that is, $L(M') = L(M)^R$. Observe, that M' is deterministic, if M was. Thus, we have shown closure of both language families under the reversal operation. \square

The above mentioned result that the family of languages accepted by double-head pushdown automata is not closed under intersection carries over to the input-driven case as well.

Theorem 11 *Both families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under intersection.*

Proof In [13] the non-closure of $\mathcal{L}(2\text{hPDA})$ under intersection was shown with the help of the 2hPDA languages $L = \{a^n b^n c^n d^m e^\ell \mid m, n, \ell \geq 1\}$ and $L' = \{a^m b^\ell c^n d^n e^n \mid n, m, \ell \geq 1\}$, since their intersection $L \cap L' = \{a^n b^n c^n d^n e^n \mid n \geq 1\}$ is not member of $\mathcal{L}(2\text{hPDA})$. Thus, in order to prove our non-closure result on intersection it suffices to show that both languages L and L' can already be accepted by a det-ID2hPDA .

We only give a brief description of a det-ID2hPDA M that accepts the language L . By a similar argumentation one can construct a det-ID2hPDA for the language L' , too. On input w the det-ID2hPDA M proceeds as follows: the right head of M moves from right to left until it reaches the first c and checks whether the input has a suffix of the $d^m e^\ell$, for some $m, \ell \geq 1$. This can be done without using the pushdown store and without moving the left head. Afterwards it again moves only its right head and pushes a C for every letter c into the pushdown store. When it reaches the first b it starts alternately moving the left and the right head, reading letter b from the right and a from the left, beginning with the right head, while it pops for every movement of the right head a C from the pushdown store. If the pushdown store is empty and the left head moves to the right, it enters an accepting state. The alphabets of M are $\Sigma_D = \{c\}, \Sigma_R = \{b\}, \Sigma_N = \{a, d, e\}$. A detailed construction of M is left to the reader. This proves the stated claim. \square

Before we continue with the complementation operation, we first establish that every deterministic and nondeterministic input-driven double-head pushdown automaton can be forced to read the entire input. This property turns out to be useful for the following construction showing the closure under complementation for deterministic input-driven double-head pushdown automata.

Lemma 12 *Let M be an ndet-ID2hPDA . Then one can construct an equivalent ndet-ID2hPDA M' , that is, $L(M') = L(M)$, that decides on acceptance/rejection after it has read the entire input. If M is deterministic, then so is M' .*

The family of languages accepted by double-head pushdown automata are not closed under complementation. We show that the family of languages accepted by deterministic input-driven double-head pushdown automata is closed under complementation, while the nondeterministic family is not closed.

Theorem 13 *The family $\mathcal{L}(\text{det-ID2hPDA})$ is closed under complementation.*

Proof Let $M = \langle Q, \Sigma, \Gamma, q_0, F, \perp, \delta_D, \delta_R, \delta_N \rangle$ be a det-ID2hPDA . By the previous lemma we can assume w.l.o.g. that M decides on acceptance/rejection after it has read the entire input. But then, if we exchange accepting and non-accepting states we accept the complement of $L(M)$. Thus, the det-ID2hPDA $M' = \langle Q, \Sigma, \Gamma, q_0, F', \perp, \delta_D, \delta_R, \delta_N \rangle$ with $F' = Q \setminus F$ is an acceptor for the language $\overline{L(M)}$. This proves our statement. \square

Since the family of languages accepted by det-ID2hPDA is not closed under intersection, it can be concluded that it is not closed under union.

Theorem 14 *The family $\mathcal{L}(\text{det-ID2hPDA})$ is not closed under union.*

Let us come back to the complementation operation. For the language family induced by ndet-ID2hPDA we obtain non-closure under complementation in contrast to the above given theorem on the deterministic language family in question.

Theorem 15 *The family $\mathcal{L}(\text{ndet-ID2hPDA})$ is not closed under complementation.*

Proof In [13] it has been shown that the language $L_1 = \{a^n b^n c^n d^n e^n \mid n \geq 1\}$ cannot be accepted even by any double-head pushdown automata. We briefly show that the language $\overline{L_1}$ is accepted by an ndet-ID2hPDA M . The complement of L_1 can be described as follows: a word is in $\overline{L_1}$ if and only if (i) it belongs to *complement* of the regular language $a^+ b^+ c^+ d^+ e^+$ or (ii) it belongs to one of the context-free languages $\{a^{n_1} b^{n_2} c^{n_3} d^{n_4} e^{n_5} \mid n_1, n_2, \dots, n_5 \geq 1 \text{ and } n_i \neq n_j, \text{ for some pair } (i, j) \in \{1, 2, \dots, 5\}^2 \text{ with } i \neq j\}$. Thus, on input w the ndet-ID2hPDA M guesses which of the above cases (i) or (ii) applies. In the first case M simulates a finite automaton without using its pushdown store. In the second case, automaton M guesses appropriate i and j with $i \neq j$ from $\{1, 2, \dots, 5\}$ and moves its two heads to the corresponding blocks of letters. Then it checks whether $n_i \neq n_j$ by alternately moving the left and right head without using the pushdown store. If $n_i \neq n_j$ and the heads meet, the automaton accepts, otherwise it rejects. Since M is not using the pushdown store at all, only the transition function δ_N is defined. Thus, the signature is $\Sigma_N = \Sigma$ and $\Sigma_D = \Sigma_R = \emptyset$.

Since M accepts $\overline{L_1}$, but L_1 cannot be accepted by any double-head pushdown automata, the language family $\mathcal{L}(\text{ndet-ID2hPDA})$ is not closed under complementation. \square

Now, L_2 can be used to show that the family of languages accepted by deterministic and nondeterministic input-driven double-head pushdown automata are not closed under concatenation and iteration.

Theorem 16 *Both language families $\mathcal{L}(\text{det-ID2hPDA})$ $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under concatenation and iteration.*

While both families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under union and intersection, they are closed under the union and intersection with regular languages.

Theorem 17 *Both families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are closed under intersection and union with regular languages.*

Next, we consider the closure under homomorphism.

Theorem 18 *Both families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under (length preserving) homomorphisms.*

Proof Consider the language $L = \{a^n \$ b^n \$ c^n \mid n \geq 1\}$. It is easy to show that L is accepted by some det-ID2hPDA with signature $\Sigma_N = \{a, \$\}$, $\Sigma_D = \{c\}$, and $\Sigma_R = \{b\}$. The details are left to the reader. Further consider the homomorphism h defined by $h(a) = a$, $h(\$) = \$$, $h(b) = a$, and $h(c) = a$ that leads to the language $h(L) = \{a^n \$ a^n \$ a^n \mid n \geq 1\}$. We show that $h(L)$ cannot be accepted by any ndet-ID2hPDA.

Assume to the contrary that there is an ndet-ID2hPDA M accepting the language $h(L)$. Observe, that the input contains only the letters a and two $\$$ s. Then we consider three cases, according to which set of the signature the letter a belongs to:

1. Letter $a \in \Sigma_N$. Then M possibly can use the pushdown store only for the two $\$$ letters. In this case, the whole computation of M can be mimicked by a finite automaton.
2. Letter $a \in \Sigma_D$. Then the pushdown of M can arbitrarily increase in height during a computation (if the a -blocks on both sides of the word are long enough), and can be decreased at most twice with the help of the letters $\$$. Again, the whole computation of M can be simulated by a finite automaton.
3. Letter $a \in \Sigma_R$. Again, the computation of M can be done by a finite state machine, since the pushdown height is bounded by two and can be stored in the finite control of an automaton. The letters a force a pop and the letters $\$$ may increase the pushdown height by at most two.

By our consideration we conclude that M can always be replaced by a finite state automaton and therefore the language $h(L)$ is regular, which is a contradiction to the pumping lemma of regular languages and to our above given assumption. Hence $h(L)$ cannot be accepted by any ndet-ID2hPDA. \square

From the Boolean operations, the union operation applied to $\mathcal{L}(\text{ndet-ID2hPDA})$ is still missing.

Theorem 19 *The family $\mathcal{L}(\text{ndet-ID2hPDA})$ is not closed under union.*

Proof Consider the language $L = \{a^n b^{2n} a^n \mid n \geq 1\}$. Using the signature $\Sigma_N = \emptyset$, $\Sigma_D = \{a\}$, and $\Sigma_R = \{b\}$ it is not hard to see that L is accepted by a det-ID2hPDA. Note that the a^n -prefix and -suffix of the input word is compared by the use of the two input heads, while the b^{2n} -infix is checked against the content of the pushdown, which is previously filled by reading the a 's from the input. Similarly, the language where the a 's and b 's are exchanged, that is, $L' = \{b^n a^{2n} b^n \mid n \geq 1\}$ is also accepted by some det-ID2hPDA.

Next, we consider the union $L \cup L'$. We show that it cannot be accepted by any ndet-ID2hPDA. Assume to the contrary that there is an ndet-ID2hPDA $M'' = \langle Q'', \{a, b\}, \Gamma, q, F, \perp, \delta_D, \delta_R, \delta_n \rangle$ that accepts the language $L \cup L'$, that is, $L(M'') = L \cup L'$. With the same argumentation as in the proof of Theorem 18 we conclude that M'' cannot accept $L \cup L'$ without using the pushdown store. Thus, one of the two input symbols a or b force M'' to push and the other symbol to pop. W.l.o.g. we assume that $\Sigma_D = \{a\}$ and $\Sigma_R = \{b\}$; the other case can be treated in a similar way. Recall that Q'' is the state set of M'' . Then consider the input word $w = b^n a^{2n} b^n$, for $n > |Q''|$. Note that while the input heads cross over the b^n -prefix and -suffix of w the automaton M'' is forced to pop from the pushdown store and thus empties it. Since $w \in L \cup L'$, there is an accepting computation of M'' on w , which is of the form

$$(q_0, b^n a^{2n} b^n, \lambda) \vdash^* (s, b^{n-i} a^{2n} b^{n-j}, \lambda) \vdash^* (s, b^{n-i-i'} a^{2n} b^{n-j-j'}, \lambda) \vdash^* (q_f, \lambda, \gamma),$$

where $s \in Q''$, $q_f \in F$, $\gamma \in \Gamma^*$, and moreover, $i + i' \leq n$ and $j + j' \leq n$ and $i' + j' \geq 1$. But then by cutting out the loop computation on the state s also the word $b^{n-i'} a^{2n} b^{n-j'}$ is accepted by M'' via the computation

$$(q_0, b^{n-i'} a^{2n} b^{n-j'}, \lambda) \vdash^* (s, b^{n-i-i'} a^{2n} b^{n-j-j'}, \lambda) \vdash^* (q_f, \lambda, \gamma).$$

Since this word is not a member of $L \cup L'$ we get a contradiction to our assumption. Therefore, the language $L \cup L'$ cannot be accepted by any ndet-ID2hPDA. Thus, the language family $\mathcal{L}(\text{ndet-ID2hPDA})$ is not closed under union. \square

For the inverse homomorphism we also get a non-closure result.

Theorem 20 *Both families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under inverse homomorphisms.*

Proof Consider the det-ID2hPDA language $L = \{a^n b^{2n} a^n \mid n \geq 1\}$ from Lemma 19. Let h be the homomorphism defined by $h(a) = a$ and $h(b) = bb$. Then $h^{-1}(L) = \{a^n b^n a^n \mid n \geq 1\}$. This is the language $L_{\alpha a}$ from Lemma 6 that cannot be accepted by any ndet-ID2hPDA. This shows that both language families $\mathcal{L}(\text{det-ID2hPDA})$ and $\mathcal{L}(\text{ndet-ID2hPDA})$ are not closed under inverse homomorphisms. \square

5 Decidability questions

In this section, we investigate the usually studied decidability questions for deterministic and nondeterministic input-driven 2hPDAs. It turns out that the results are similar to those obtained for conventional deterministic and nondeterministic pushdown automata. In particular, we obtain the decidability of emptiness and finiteness for det-ID2hPDAs and ndet-ID2hPDAs as well as the decidability of equivalence with a regular set, inclusion in a regular set, and inclusion of a regular set. On the other hand, inclusion turns out to be not even semidecidable for det-ID2hPDAs and universality, equivalence, and regularity are not semidecidable for ndet-ID2hPDA as well. Finally, the decidability and non-semidecidability results can be translated to hold for double-ID2hPDAs correspondingly.

Theorem 21 *Let M be an ndet-2hPDA. Then, it is decidable whether or not $L(M)$ is empty or finite.*

Proof In [13] Nagy shows that for every ndet-2hPDA a classical NPDA accepting a letter-equivalent context-free language can effectively be constructed. Thus, the emptiness and finiteness problems for an ndet-2hPDA can be reduced to the corresponding problems for an NPDA which are known to be decidable. \square

Corollary 22 *Let M be an ndet-ID2hPDA or det-ID2hPDA. Then, it is decidable whether or not $L(M)$ is empty or finite.*

To obtain undecidability results we will use the technique of *valid computations* of Turing machines which is presented, for example, in [8]. This technique allows to show that some questions are not only undecidable, but moreover not semidecidable, where we say that a problem is *semidecidable* if and only if the set of all instances for which the answer is “yes” is recursively enumerable (see, for example, [8]). Let $M = \langle Q, \Sigma, T, \delta, q_0, B, F \rangle$ be a deterministic Turing machine, where T is the set of tape symbols including the set of input symbols Σ and the blank symbol B , Q is the finite set of states and $F \subseteq Q$ is the set of final states. The initial state is q_0 and δ is the transition function. Without loss of generality, we assume that Turing machines can halt only after an odd number of moves, halt whenever they enter an accepting state, make at least three moves, and cannot print blanks. At any instant during a computation, M can be completely described by an *instantaneous description* (ID) which is a string $tqt' \in T^*QT^*$ with the following meaning: M is in the state q , the non-blank tape content is the string tt' , and the head scans the first symbol of t' . The initial ID of M on input $x \in \Sigma^*$ is $w_0 = q_0x$. An ID is accepting whenever it belongs to T^*FT^* . The set $\text{VALC}(M)$ of valid (accepting) computations of M consists of all finite strings of the form $w_0\#w_2\#\cdots\#w_{2n}\$w_{2n+1}^R\#\cdots\#w_3^R\#w_1^R$ where $\#, \$ \notin T \cup Q$, w_i , $0 \leq i \leq 2n+1$, are instantaneous description of M , w_0 is an initial ID, w_{2n+1} is an accepting (hence halting) configuration, w_{i+1} is the successor configuration of w_i , $0 \leq i \leq 2n$. The set of *invalid computations* $\text{INVALC}(M)$ is the complement of $\text{VALC}(M)$ with respect to the alphabet $T \cup Q \cup \{\#, \$\}$.

Theorem 23 *Let M_1 and M_2 be two det-ID2hPDAs. Then, the question $L(M_1) \cap L(M_2) = \emptyset$ is not semidecidable.*

Proof We will first show that the set of valid computation $\text{VALC}(M)$ of a Turing machine M is the intersection of two languages L_1 and L_2 where each language is accepted by some det-ID2hPDA. We define L_1 to consist of all strings of the form $w_0\#w_2\#\cdots\#w_{2n}\$w_{2n+1}^R\#\cdots\#w_3^R\#w_1^R$, where w_{i+1} is the successor configuration of w_i for all even $0 \leq i \leq 2n$. Language L_2 is defined as the set of all strings of the form $w_0\#w_2\#\cdots\#w_{2n}\$w_{2n+1}^R\#\cdots\#w_3^R\#w_1^R$, where w_{i+1} is the successor configuration of w_i for

all odd $1 \leq i \leq 2n - 1$. Moreover, w_0 is an initial ID and w_{2n+1} is an accepting ID. It is clear that $L_1 \cap L_2 = \text{VALC}(M)$. Next, we sketch how L_1 can be accepted by some det-ID2hPDA M_1 . The partition of the input alphabet is $\Sigma_D = \Sigma_R = \emptyset$ and $\Sigma_N = T \cup Q \cup \{\#, \$\}$. Thus, we will not make use of the pushdown store in our construction. The basic idea is that both heads of M_1 move successively to the right resp. left checking that the ID that is seen by the right head is indeed the successor configuration seen by the left head. This is possible since the changes between a configuration and its successor configuration are only local and hence can be checked using the state set of M_1 . Moreover, the state set is also used to check the correct format of the input, where the left head checks the input part to the left of the marker $\$$, whereas the right head checks the input part to the right of $\$$. The computation ends accepting when both heads meet at the marker $\$$ and all previous checks have been successful. A det-ID2hPDA M_2 for L_2 works similarly. First, the left head has to skip the initial ID w_0 . Then, both heads of M_2 move successively to the right resp. left checking that the ID that is seen by the left head is indeed the successor configuration seen by the right head. Again, the correct format of the input is implicitly checked. When the left head has reached the marker $\$$, the right head has to skip the accepting ID w_{2n+1} and the computation ends accepting when both heads meet at the marker $\$$ and all previous checks have been successful. Since $L_1 \cap L_2 = \text{VALC}(M)$ and the emptiness problem for Turing machines is not semidecidable (see, for example, [8]), the claim of the theorem follows. \square

Since $\mathcal{L}(\text{det-ID2hPDA})$ is closed under complementation owing to Theorem 13, we immediately obtain that the inclusion problem is not semidecidable.

Corollary 24 *Let M_1 and M_2 be two det-ID2hPDAs. Then, it is not semidecidable whether or not $L(M_1) \subseteq L(M_2)$.*

However, in case of regular languages we can decide inclusion and equivalence.

Theorem 25 *Let M be a det-ID2hPDA and R be a regular language. Then, it is decidable whether or not $L(M) = R$, $R \subseteq L(M)$, or $L(M) \subseteq R$.*

Proof First, we note that $R \subseteq L(M)$ if and only if $R \cap \overline{L(M)} = \emptyset$ and that $L(M) \subseteq R$ if and only if $L(M) \cap \overline{R} = \emptyset$. Since $\mathcal{L}(\text{det-ID2hPDA})$ is closed under complementation and under intersection with regular languages by Theorem 13 and Theorem 17, the regular languages are closed under complementation, and emptiness is decidable for det-ID2hPDAs owing to Theorem 21, all claims of the theorem follow. \square

The decidability of the latter questions gets lost if the given ID2hPDA is nondeterministic, since in this case even the universality question is not semidecidable.

Theorem 26 *Let M be an ndet-ID2hPDA. Then, the questions of universality, equivalence, and regularity are not semidecidable.*

Owing to Theorem 21 it is clear that emptiness and finiteness are decidable for det-double-ID2hPDAs and ndet-double-ID2hPDAs as well. Since the language family accepted by det-double-ID2hPDAs is also closed under complementation and intersection with regular languages, we obtain that decidable questions for det-ID2hPDAs are also decidable for det-double-ID2hPDAs. On the other hand, the non-semidecidability results obtained for ID2hPDAs in the single mode obviously hold for the double mode as well. It is currently an open problem whether equivalence and regularity are decidable for det-ID2hPDAs or det-double-ID2hPDAs, whereas both problems are known to be decidable for DPDAs.

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