

From Linear Term Rewriting to Graph Rewriting with Preservation of Termination

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Encodings of term rewriting systems (TRSs) into graph rewriting systems usually lose global termination, meaning the encodings do not terminate on all graphs. A typical encoding of the terminating TRS rule $a(b(x)) \rightarrow b(a(x))$, for example, may be indefinitely applicable along a cycle of a 's and b 's. Recently, we introduced PBPO⁺, a graph rewriting formalism in which rules employ a type graph to specify transformations and control rule applicability. In the present paper, we show that PBPO⁺ allows for a natural encoding of linear TRS rules that preserves termination globally. This result is a step towards modeling other rewriting formalisms, such as lambda calculus and higher order rewriting, using graph rewriting in a way that preserves properties like termination and confluence. We moreover expect that the encoding can serve as a guide for lifting TRS termination methods to PBPO⁺ rewriting.

1 Introduction

A *rewriting framework* \mathcal{F} consists of a set of *objects* O and a set of *rewriting systems* \mathcal{R} . Each system $R \in \mathcal{R}$ is a set of *rewrite rules*. Each rule $\rho \in R$ defines a particular *rewrite relation* $\rightarrow_\rho \subseteq O \times O$ on objects, and the rules of R collectively give rise to a general rewrite relation $\rightarrow_R = \bigcup_{\rho \in R} \rightarrow_\rho$. The usual definitions of string, cycle and term rewriting systems (TRSs), and the various definitions of term graph and graph rewriting formalisms, are instances of this abstract view.

Because terms can be viewed as generalizations of strings, term graphs as generalizations of terms, graphs as generalizations of terms graphs and cycles, etc., the question whether one framework can be encoded into another framework frequently arises naturally. The same is true when comparing the large variety of graph rewriting frameworks. Moreover, the properties such an encoding is expected to satisfy may vary. Let us therefore fix some vocabulary.

Definition 1 (Encoding). An *encoding* \mathcal{E} of a framework \mathcal{F} into a framework \mathcal{G} consists of a function $\mathcal{E}_O : O^{\mathcal{F}} \rightarrow O^{\mathcal{G}}$ on objects and a function $\mathcal{E}_{\mathcal{R}} : \mathcal{R}^{\mathcal{F}} \rightarrow \mathcal{R}^{\mathcal{G}}$ on rewrite systems. The subscript is usually omitted, since it will be clear from context which of \mathcal{E}_O and $\mathcal{E}_{\mathcal{R}}$ is meant.

Given an encoding \mathcal{E} , a variety of properties of interest may be distinguished. We will say \mathcal{E} is

1. *step-preserving* if $x \rightarrow_R^{\mathcal{F}} y \implies \mathcal{E}(x) \rightarrow_{\mathcal{E}(R)}^{\mathcal{G}} \mathcal{E}(y)$;
2. *closed* if $x \rightarrow_{\mathcal{E}(R)}^{\mathcal{G}} y$ and $x \cong \mathcal{E}(x')$ for an $x' \in O^{\mathcal{F}} \implies y \cong \mathcal{E}(y')$ for some $y' \in O^{\mathcal{F}}$ with $x' \rightarrow_R y'$;
3. an *embedding* if \mathcal{E} is step-preserving and closed;
4. *globally P-preserving* (for a property P , such as termination or confluence), if whenever $R \in \mathcal{R}^{\mathcal{F}}$ satisfies P , then so does the system $\mathcal{E}(R) \in \mathcal{R}^{\mathcal{G}}$ on all objects $O^{\mathcal{G}}$; and
5. *locally P-preserving* if whenever $R \in \mathcal{R}^{\mathcal{F}}$ satisfies P , then so does the system $\mathcal{E}(R) \in \mathcal{R}^{\mathcal{G}}$ on the restricted domain of objects $\mathcal{E}(O^{\mathcal{F}}) \subseteq O^{\mathcal{G}}$.

Consider the string rewrite rule $ab \rightarrow ba$ and its usual encoding $a(b(x)) \rightarrow b(a(x))$ as a term rewrite rule. This encoding is an embedding that preserves termination and confluence globally. The usual encoding as a cycle rewrite rule, by contrast, is step-preserving, but not closed, and neither termination- nor confluence-preserving.

Building on PBPO by Corradini et al. [4] and our own patch graph rewriting formalism [16], we recently proposed the PBPO⁺ algebraic graph rewriting approach [17], in which rules employ a type graph to specify transformations and control rule applicability. In the present paper we give an embedding of linear term rewrite systems into PBPO⁺ that preserves global termination, despite being applicable to graphs that are not encodings of terms. This result requires powerful features (unsupported by standard approaches such as DPO [8]), as two examples illustrate:

1. For the encoding of $f(x,y) \rightarrow f(a,y)$ to be step-preserving, it must be possible to delete an arbitrary subgraph x below f , while leaving the context above of f and the subgraph corresponding to y intact.
2. For the encoding of $a(b(x)) \rightarrow b(a(x))$ to be terminating, the rule must not be applicable on a cycle of a 's and b 's.

Apart from being an interesting expressiveness result for PBPO⁺, our result enables reduction-style termination arguments for linear, ‘term-like’ PBPO⁺ rewrite rules. Moreover, as we will elaborate in the discussion (Section 6), we believe our result has broader relevance for the development of termination techniques for graph rewriting, as well as the modeling of other rewrite formalisms.

The structure of the paper is as follows. In Section 2, we summarize the relevant categorical and TRS preliminaries. In Section 3, we give a self-contained introduction to PBPO⁺ and to **Graph**^(\mathcal{L}, \leq) [17], a special category that combines well with PBPO⁺. In Section 4, we define an embedding of linear term rewriting into PBPO⁺ rewriting over category **Graph**^(\mathcal{L}, \leq). In Section 5, we prove that the embedding is globally termination-preserving, using a novel zoning proof. Finally, we discuss the significance of our results in Section 6.

2 Preliminaries

We assume familiarity with various basic categorical notions, notations and results, including morphisms $X \rightarrow Y$, pullbacks and pushouts, monomorphisms (monos) $X \twoheadrightarrow Y$ (note the different arrow notation) and identities $1_X : X \twoheadrightarrow X$ [1, 14].

Definition 2 (Graph Notions). A (labeled) *graph* G consists of a set of vertices V , a set of edges E , source and target functions $s, t : E \rightarrow V$, and label functions $\ell^V : V \rightarrow \mathcal{L}$ and $\ell^E : E \rightarrow \mathcal{L}$ for some label set \mathcal{L} . A graph is *unlabeled* if \mathcal{L} is a singleton.

A *premorph* between graphs G and G' is a pair of maps $\phi = (\phi_V : V_G \rightarrow V_{G'}, \phi_E : E_G \rightarrow E_{G'})$ satisfying $(s_{G'}, t_{G'}) \circ \phi_E = \phi_V \circ (s_G, t_G)$.

A *homomorphism* is a label-preserving premorph ϕ , i.e., a premorph satisfying $\ell_{G'}^V \circ \phi_V = \ell_G^V$ and $\ell_{G'}^E \circ \phi_E = \ell_G^E$.

Definition 3 (Category **Graph** [7]). The category **Graph** has graphs as objects, parameterized over some global (and usually implicit) label set \mathcal{L} , and homomorphisms as arrows. **FinGraph** is the full subcategory of finite graphs.

The following TRS definitions are all standard [18].

Definition 4 (Signature). A *signature* Σ consists of a non-empty set of *function symbols* $f, g, \dots \in \Sigma$, equipped with an *arity function* $\# : \Sigma \rightarrow \mathbb{N}$. Nullary function symbols a, b, \dots are called *constants*.

Definition 5 (Terms). The set of *terms* $\mathbf{l}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \dots \in \text{Ter}(\Sigma, \mathcal{X})$ over a signature Σ and an infinite set of variables $x, y, \dots \in \mathcal{X}$ is defined inductively by:

- $x \in \text{Ter}(\Sigma, \mathcal{X})$ for every $x \in \mathcal{X}$;
- if $f \in \Sigma$ with $\#(f) = n$, and $\mathbf{t}_1, \dots, \mathbf{t}_n \in \text{Ter}(\Sigma, \mathcal{X})$, then $f(\mathbf{t}_1, \dots, \mathbf{t}_n) \in \text{Ter}(\Sigma, \mathcal{X})$. If $n = 0$, we write f instead of $f()$.

A term \mathbf{t} is *linear* if every $x \in \mathcal{X}$ occurs at most once in \mathbf{t} . We write $\text{Var}(\mathbf{t})$ to denote the set of variables occurring in \mathbf{t} .

Definition 6 (Position). A *position* p is a sequence of integers, i.e., $p \in \mathbb{N}^*$. The empty sequence is denoted by ε . We write pn (and np) to denote the right (and left) concatenation of a positive integer n to a position p .

Every symbol occurrence in a term has a position associated with it. The position of the head symbol is ε , and the position of the i -th ($i \geq 1$) symbol below a symbol with position p is pi . For a term \mathbf{s} and a position p in \mathbf{s} , we write $\mathbf{s}(p)$ to denote the symbol at position p in \mathbf{s} .

Definition 7 (Substitutions). A *substitution* is a function $\sigma : \mathcal{X} \rightarrow \text{Ter}(\Sigma, \mathcal{X})$. For terms $\mathbf{s} \in \text{Ter}(\Sigma, \mathcal{X})$ we define $\mathbf{s}\sigma \in \text{Ter}(\Sigma, \mathcal{X})$ by $x\sigma = \sigma(x)$ for $x \in \mathcal{X}$, and $f(\mathbf{t}_1, \dots, \mathbf{t}_n)\sigma = f(\mathbf{t}_1\sigma, \dots, \mathbf{t}_n\sigma)$ for $f \in \Sigma$ and $\mathbf{t}_1, \dots, \mathbf{t}_n \in \text{Ter}(\Sigma, \mathcal{X})$.

Definition 8 (Contexts). A *context* $C[\]$ is a term from $\text{Ter}(\Sigma, \mathcal{X} \uplus \{\square\})$ with exactly one occurrence of the *hole* \square . We write $C[\mathbf{t}]$ for the term obtained by replacing the hole with \mathbf{t} .

Definition 9 (Term Rewriting Systems). A *term rewrite rule* is a pair of terms $\mathbf{l} \rightarrow \mathbf{r}$ satisfying $\mathbf{l} \notin \mathcal{X}$ and $\text{Var}(\mathbf{r}) \subseteq \text{Var}(\mathbf{l})$. The rule is *linear* if both terms \mathbf{l} and \mathbf{r} are linear. A *term rewriting system* (TRS) \mathcal{R} is a set of term rewrite rules. The system \mathcal{R} is linear if all its rules are.

A TRS \mathcal{R} induces a relation \rightarrow on $\text{Ter}(\Sigma, \mathcal{X})$, the *rewrite relation* of \mathcal{R} , as follows: $C[\mathbf{l}\sigma] \rightarrow C[\mathbf{r}\sigma]$ for every context C , substitution σ and rule $\mathbf{l} \rightarrow \mathbf{r} \in \mathcal{R}$. The *rewrite step* $C[\mathbf{l}\sigma] \rightarrow C[\mathbf{r}\sigma]$ is said to be an *application of the rule* $\rho = \mathbf{l} \rightarrow \mathbf{r}$ *at position* p , where p is the position of the hole in $C[\]$.

3 PBPO⁺ and Graph^(\mathcal{L}, \leq)

We recently introduced PBPO⁺ [17] (short for *PBPO with strong matching*), an algebraic rewriting formalism obtained by strengthening the matching mechanism of PBPO by Corradini et al. [4]. We believe PBPO⁺ is of interest for at least three important reasons.

First, PBPO⁺ is expressive: for **Graph** in particular, and assuming monic matching, we conjecture [17] that PBPO⁺ is able to faithfully model DPO, SPO [13], SqPO [5], AGREE [3] and PBPO. More precisely, for any rule in such a formalism, there exists a PBPO⁺ rule that generates exactly the same rewrite relation.

Second, PBPO⁺ makes relatively weak assumptions on the underlying category: it is sufficient to require the existence of pushouts along monomorphisms and the existence of pullbacks. In particular, adhesivity [12], assumed for DPO rewriting to ensure the uniqueness of pushout complements, is not required.

Third, we have defined a non-adhesive category called **Graph**^(\mathcal{L}, \leq) [17] that combines very nicely with PBPO⁺, allowing graph rewrite rules to easily model notions of relabeling, type systems, wildcards and variables. These notions have been significantly more challenging to define for DPO.

In this section we provide the necessary background on PBPO⁺ and **Graph**^(\mathcal{L}, \leq).

Definition 10 (PBPO⁺ Rewriting [17]). A PBPO⁺ rewrite rule ρ (left) and adherence morphism $\alpha : G_L \rightarrow L'$ induce a rewrite step $G_L \Rightarrow_{\rho}^{\alpha} G_R$ on arbitrary G_L and G_R if the properties indicated by the commuting diagram on the right hold

$$\rho = \begin{array}{ccc} L & \xleftarrow{l} K & \xrightarrow{r} R \\ t_L \downarrow & \text{PB} \downarrow & t_K \downarrow \\ L' & \xleftarrow{l'} K' & \end{array} \quad \begin{array}{ccccc} & & K & \xrightarrow{r} & R \\ & & \downarrow !u & \swarrow & \downarrow \text{PO} \\ L & \xrightarrow{m} & G_L & \xleftarrow{g_L} & G_K & \xrightarrow{g_R} & G_R \\ 1_L \downarrow & \text{PB} \downarrow & \alpha \downarrow & & \text{PB} \downarrow & u' \downarrow & t_K \downarrow \\ L & \xrightarrow{t_L} & L' & \xleftarrow{l'} & K' & \end{array}$$

where $u : K \rightarrow G_K$ is the unique mono satisfying $t_K = u' \circ u$ [17, Lemma 11]. We write $G_L \Rightarrow_{\rho} G_R$ if $G_L \Rightarrow_{\rho}^{\alpha} G_R$ for some α .

In the rewrite rule diagram, L is the *lhs pattern* of the rule, L' its *type graph* and t_L the *typing* of L . Similarly for the *interface* K . R is the *rhs pattern* or *replacement* for L . The rewrite step diagram can be thought of as consisting of a match square (modeling an application condition), a pullback square for extracting (and possibly duplicating) parts of G_L , and finally a pushout square for gluing these parts along pattern R . The inclusion of the match square is the main aspect which differentiates PBPO⁺ from PBPO: intuitively, it prevents α from collapsing context elements of G_L onto the pattern $t_L(L) \subseteq L'$.

For the present paper, it suffices to restrict attention to rules in which l' does not duplicate subgraphs.

Definition 11 (Linear PBPO⁺ Rule). A PBPO⁺ rule is *linear* if the morphism $l' : K' \rightarrow L'$ is monic.

Remark 12. For linear PBPO⁺ rewriting, it is enough to assume the existence of pushouts and pullbacks along monomorphisms. An interesting question is whether these weakened requirements enable new use cases.

The category $\mathbf{Graph}^{(\mathcal{L}, \leq)}$ is similar to \mathbf{Graph} . The difference is that it is assumed that the label set forms a complete lattice, and that morphisms do not decrease labels. The complete lattice requirement ensures that pushouts and pullbacks are well-defined.

Definition 13 (Complete Lattice). A *complete lattice* (\mathcal{L}, \leq) is a poset such that all subsets S of \mathcal{L} have a supremum (join) $\bigvee S$ and an infimum (meet) $\bigwedge S$.

Definition 14 (Category $\mathbf{Graph}^{(\mathcal{L}, \leq)}$ [17]). For a complete lattice (\mathcal{L}, \leq) , the category $\mathbf{Graph}^{(\mathcal{L}, \leq)}$ is the category in which objects are graphs are labeled from \mathcal{L} , and arrows are graph premorphisms $\phi : G \rightarrow G'$ that satisfy $\ell_G(x) \leq \ell_{G'}(\phi(x))$ for all $x \in V_G \cup E_G$. We let $\mathbf{FinGraph}^{(\mathcal{L}, \leq)}$ denote the full subcategory of finite graphs.

Proposition 15. In $\mathbf{Graph}^{(\mathcal{L}, \leq)}$, monomorphisms are stable under pushout.

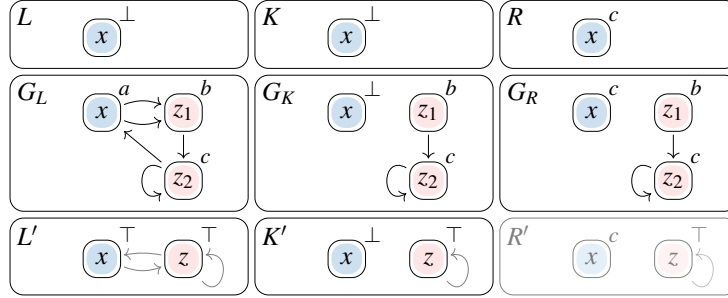
Proof. Assume given a span $B \xleftarrow{b} A \xrightarrow{c} C$ in $\mathbf{Graph}^{(\mathcal{L}, \leq)}$. Overloading names, consider the unlabeled version in \mathbf{Graph} , and construct the pushout $B \xrightarrow{m} D \xleftarrow{n} C$. Morphism $m : B \rightarrow D$ is monic, because monos are stable in the category of unlabeled graphs, by virtue of it being an adhesive category. Now for each $x \in V_D \cup E_D$, define the label function $\ell(x)$ to be the supremum of all labels in the labeled preimages $m^{-1}(x)$ and $n^{-1}(x)$, and define the $\mathbf{Graph}^{(\mathcal{L}, \leq)}$ object $D_{\ell} = (V_D, E_D, s_D, t_D, \ell)$. Then it is easy to verify that $B \xrightarrow{m} D_{\ell} \xleftarrow{n} C$ is the pushout of $B \xleftarrow{b} A \xrightarrow{c} C$ in $\mathbf{Graph}^{(\mathcal{L}, \leq)}$. \square

In this paper we will use the following simple complete lattice only.

Definition 16 (Flat Lattice [17]). Let $\mathcal{L}^{\perp, \top} = \mathcal{L} \uplus \{\perp, \top\}$. We define the *flat lattice* induced by \mathcal{L} as the poset $(\mathcal{L}^{\perp, \top}, \leq)$, which has \perp as a global minimum and \top as a global maximum, and where all elements of \mathcal{L} are incomparable. In this context, we refer to \mathcal{L} as the *base label set*.

The following example is a variation of an example found in our previous paper [17, Example 40]. It exemplifies all relevant features of linear PBPO⁺ rewriting in category $\mathbf{Graph}^{(\mathcal{L}, \leq)}$.

Example 17 (Rewrite Example). As vertex labels we employ the flat lattice induced by the base label set $\{a, b, c, \dots\}$, and we assume edges are unlabeled for notational simplicity. The diagram



displays a rule (L, L', K, K', R) which

- matches an arbitrarily labeled, loopless node x , in an arbitrary context;
- “hard overwrites” the label of x to label c ;
- disconnects x from its component by deleting its incident edges; and
- leaves all other nodes, edges and labels unchanged.

The pushout $K' \xrightarrow{r'} R' \xleftarrow{r} R$ for $\text{span } K' \xleftarrow{r} K \xrightarrow{r} R$ is depicted as well (in lower opacity), because it shows the schematic effect of applying the rewrite rule. An application to a host graph G_L is included in the middle row.

With respect to the labeling, the example demonstrates how (i) labels in L serve as lower bounds for matching, (ii) labels in L' serve as upper bounds for matching, (iii) labels in K' can be used to decrease matched labels (so in particular, \perp “instructs” to “erase” the label and overwrite it with \perp , and \top “instructs” to preserve labels), and (iv) labels in R can be used to increase labels.

4 Embedding Linear Term Rewriting Systems

We are now ready to define an encoding (Definition 1) of linear term rewrite systems into PBPO⁺. We also show that the encoding is an embedding (Theorem 35). In the next section, we prove that the embedding is globally termination-preserving.

For defining the encoding of terms as graphs, the auxiliary notion of a rooted graph is convenient.

Definition 18 (Rooted Graph). A *rooted graph* (G, r) consists of a graph G and a distinguished *root* $r \in V_G$. We let $\text{graph}((G, r)) = G$ and $\text{root}((G, r)) = r$.

We usually omit $\text{graph}(\dots)$ in places where a non-rooted graph is expected, since confusion is unlikely to occur. In visual depictions, the root r is highlighted in a circle \textcircled{r} .

Definition 19 (Term Encoding). Define the flat lattice Σ° for signatures Σ by $\Sigma^\circ = (\Sigma \uplus \mathbb{N}^+)^{\perp, \top}$.

For linear terms $\mathbf{t} \in \text{Ter}(\Sigma, \mathcal{X})$, we define the *term encoding* \mathbf{t}° of \mathbf{t} as the Σ° -labeled rooted graph $\mathbf{t}^\circ = \mathcal{E}(\mathbf{t}, \varepsilon)$, where $\mathcal{E}(\mathbf{t}, p)$ is defined by clauses

$$\mathcal{E}(f(t_1, \dots, t_n), p) = \begin{array}{c} \textcircled{p^f} \\ \swarrow \quad \searrow \\ \mathcal{E}(t_1, p_1) \quad \dots \quad \mathcal{E}(t_n, p_n) \end{array}$$

and $\mathcal{E}(x, p) = (x^\perp, x)$ for $f \in \Sigma$, $\mathbf{t}_1, \dots, \mathbf{t}_n \in \text{Ter}(\Sigma, \mathcal{X})$, $x \in \mathcal{X}$ and $p \in \mathbb{N}^*$. The target of an edge pointing towards a rooted graph (G, p') is p' . In these graphs, the identity of an edge with source p and target p' is (p, p') .

Note that the term encoding always results in a tree, because the terms it operates on are linear.

Definition 20 (Positions in Term Encodings). Analogous to positions in terms \mathbf{t} (Definition 6), we assign positions to the nodes of \mathbf{t}° : $\text{root}(\mathbf{t}^\circ)$ is assigned position ε ; and if $v \xrightarrow{i} w$ is an edge in $E_{\mathbf{t}^\circ}$ (for $i \geq 1$) and v has position p , then w is assigned position pi .

A translated rule ρ° is said to be *applied at position p in \mathbf{t}°* if the match morphism $m : L \rightarrow \mathbf{t}^\circ$ maps the root of L onto the vertex with position p in \mathbf{t}° , and establishes a match.

The following definition is used in the setting of rule encodings.

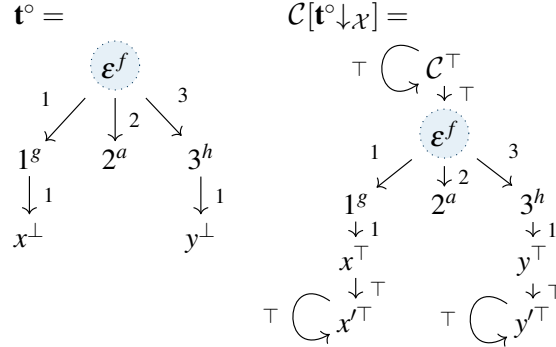
Definition 21 (Context Closures). Let $G_r = (G, r)$ be a rooted graph.

Assume $\mathcal{C} \notin V_G$. The *upper context closure* of G_r , denoted $\mathcal{C}[G_r]$, is the r -rooted graph obtained by adding a \top -labeled vertex \mathcal{C} and two \top -labeled edges with identities (\mathcal{C}, r) and $(\mathcal{C}, \mathcal{C})$ to G . Sources and targets are given by the first and second projections, respectively.

For $x \in \mathcal{X}$, let x' be fresh for V_G . The *lower context closure* of G w.r.t. a subset $\mathcal{X} \subseteq V_G$, denoted $G \downarrow_{\mathcal{X}}$, is the r -rooted graph obtained as follows: for every $x \in V_G \cap \mathcal{X}$, (i) relabel x to \top , and (ii) add a \top -labeled vertex x' and two \top -labeled edges (x, x') and (x', x') to G .

The *context closure* of G_r is defined as $\mathcal{C}[G_r \downarrow_{\mathcal{X}}]$.

Example 22. The term encoding \mathbf{t}° of $\mathbf{t} = f(g(x), a, h(y))$ and its context closure $\mathcal{C}[\mathbf{t}^\circ \downarrow_{\mathcal{X}}]$ are shown on the right. Both graphs are rooted in ε . (The edge identities are left implicit.)



Definition 23 (Variable Heads and Symbol Vertices). For term encodings \mathbf{t}° , the vertices in $x \in V_{\mathbf{t}^\circ} \cap \mathcal{X}$ with $\ell(x) = \perp$ are called *variable heads*, and the remaining vertices labeled from Σ are called *symbol vertices*.

Definition 24 (Interface Graph). The *interface graph* $\mathcal{I}(\mathbf{t})$ for a term \mathbf{t} is the rooted graph (G', ε) , where G' is the discrete graph induced by $V_{G'} = \text{Var}(\mathbf{t}) \cup \{\varepsilon\}$ and $\ell_{G'}(v) = \perp$ for all $v \in V_{G'}$.

Definition 25 (Rule Encoding). The *rule encoding* ρ° of a linear term rewrite rule $\rho : \mathbf{l} \rightarrow \mathbf{r}$ over Σ into a (linear) PBPO⁺ rewrite rule over Σ° -labeled graphs is defined as follows:

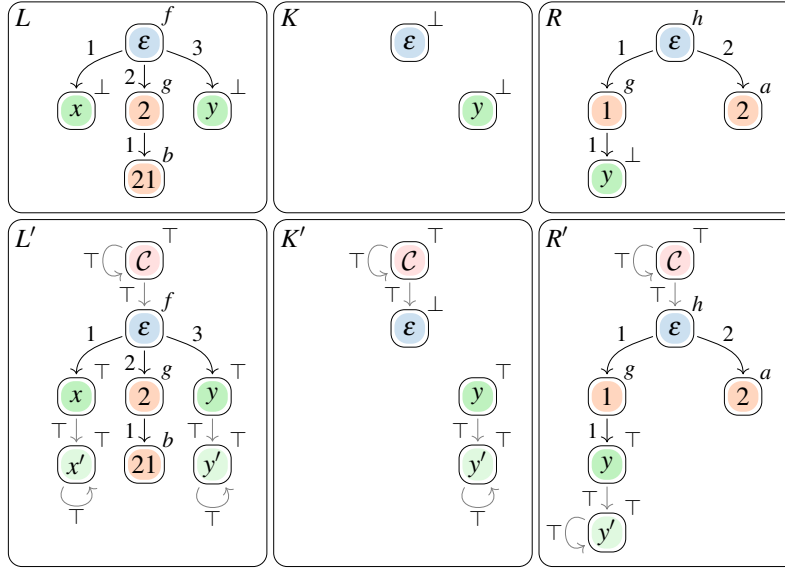
$$\begin{array}{lll} L = \mathbf{l}^\circ & K = \mathcal{I}(\mathbf{r}) & R = \mathbf{r}^\circ \\ L' = \mathcal{C}[\mathbf{l}^\circ \downarrow_{\mathcal{X}}] & K' = \mathcal{C}[\mathcal{I}(\mathbf{r}) \downarrow_{\mathcal{X}}] & \end{array}$$

Here we implicitly consider the rooted graphs as graphs by forgetting their roots. Each of the morphisms l, r, l', t_L , and t_K map roots to roots and behave as inclusions otherwise.

Observe that the rule encoding accounts for the special case where the right-hand side \mathbf{r} of the TRS rule is a variable x , in which case $r : K \rightarrow R$ is the morphism determined by $r(\varepsilon) = r(x) = x$. (The case where the left-hand side \mathbf{l} is a variable is excluded by definition.)

Proposition 26. *In Definition 25, all of the morphisms are well-defined and uniquely determined, and the pullback property is satisfied. Moreover, morphisms l, l', t_L and t_K are monic, and r is monic iff \mathbf{r} is not a variable.* \square

Example 27 (Rule Encoding). The TRS rule $\rho = f(x, g(b), y) \rightarrow h(g(y), a)$ is encoded as the PBPO⁺ rewrite rule ρ° given by



An application of this rule can be thought of as binding head variable x and y of L to the roots of two subterms. These subterms and the context are then uniquely captured by L' (by virtue of the strong match property), and correctly rearranged around R by the rewrite step.

Rule encodings extend to rewrite system encodings in the obvious way.

Definition 28 (Rewrite System Encoding). The *rewrite system encoding* R° of a linear TRS R is $\{\rho^\circ \mid \rho \in R\}$.

All the encodings we have introduced have obvious inverses.

Definition 29 (Decoding). For term/rule/system encodings x° , we define the inverse $decode(x^\circ) = x$.

Proposition 30 (Root Mapping Determines Adherence). *Let $\rho = \mathbf{l} \rightarrow \mathbf{r}$ be a linear term rewrite rule. If ρ° is applied at position p in \mathbf{s}° , then a unique $\alpha : \mathbf{s}^\circ \rightarrow \mathcal{C}[\mathbf{l}^\circ \downarrow_{\mathcal{X}}]$ exists that establishes a strong match, i.e., that makes*

$$\begin{array}{ccc}
 \mathbf{l}^\circ & \xrightarrow{m} & \mathbf{s}^\circ \\
 \downarrow \mathbf{1}_{\mathbf{l}^\circ} & \lrcorner PB & \downarrow \alpha \\
 \mathbf{l}^\circ & \xrightarrow{t_L} & \mathcal{C}[\mathbf{l}^\circ \downarrow_{\mathcal{X}}]
 \end{array}$$

a pullback square.

Proof. By definition of applying at a position p , m maps $root(\mathbf{l}^\circ)$ onto position p of \mathbf{s}° , fully determining m to map nodes with position q in \mathbf{l}° onto nodes with position pq in \mathbf{s}° . A node in \mathbf{l}° is either a symbol vertex or a variable head. For symbol vertices, any m must preserve labels. Variable heads (labeled with \perp) are mapped by m onto either (i) vertex $\sigma(x)$ labeled with \perp if $\sigma(x) \in \mathcal{X}$ is a variable, or (ii) vertex pq labeled with $f \in \Sigma$ if x is substituted for some non-variable term $\sigma(x) = f(\mathbf{t}_1, \dots, \mathbf{t}_n)$ ($n \geq 0$).

On the image $m(\mathbf{I}^\circ)$, define α such that $t_L = \alpha \circ m$. The labels of symbol vertices are thereby preserved, and the labels in the head variables of \mathbf{I}° are increased to \top . The elements not in $m(\mathbf{I}^\circ)$ can be mapped onto the appropriate elements added by the context closure, and only in one way as to not overlap with t_L . Because t_L does not map onto these closures, pulling α along t_L gives the required pullback square. \square

Lemma 31 (Match Determinism). *Let $\rho = \mathbf{l} \rightarrow \mathbf{r}$ be a linear term rewrite rule. If ρ° is applied at position p in \mathbf{s}° and gives rise to a step $\mathbf{s}^\circ \rightarrow G$, then G is uniquely determined up to isomorphism.*

Proof. By Proposition 30, adherence α is completely determined, and by general categorical properties, the pullback of α along l' gives a unique result up to isomorphism, and so does the final pushout. \square

Proposition 32. *If $m : \mathbf{I}^\circ \rightarrow \mathbf{s}^\circ$ is a mono, then $\mathbf{s}^\circ = (C[\mathbf{l}\sigma])^\circ$ for some context C and substitution σ . Moreover, the position of $m(\text{root}(\mathbf{I}^\circ))$ in \mathbf{s}° equals the position of \square in $C[\]$.*

Proof. By monicity of m , the tree structure of \mathbf{I}° is preserved into \mathbf{s}° . The labels of symbol vertices and edges are also preserved, since \mathbf{s}° has no occurrences of \top . This also means that, for every vertex v of \mathbf{I}° , v and $m(v)$ have the same number of outgoing edges, since encodings preserve arities.

A variable head $x \in V_{\mathbf{I}^\circ}$ is mapped onto a vertex $m(x)$, which is either a variable head with label \perp , or a symbol vertex labeled with some $f \in \Sigma$ and a subtree underneath.

Let p be the position of $m(\text{root}(\mathbf{I}^\circ))$ in \mathbf{s}° . Define C as the context obtained from \mathbf{s} by replacing the subterm at position p by \square . Define the substitution σ , for every $x \in \text{Var}(\mathbf{l})$, by $\sigma(x) = \mathbf{s}|_{pq_x}$ where q_x is the position of x in \mathbf{l} . Then the claim follows since m maps x in \mathbf{I}° to the position pq_x in \mathbf{s}° , and the subtree rooted at this position is $(\mathbf{s}|_{pq_x})^\circ$. \square

Lemma 33 ($(\cdot)^\circ$ Is Step-Preserving). *Let $\rho = \mathbf{l} \rightarrow \mathbf{r}$ be a linear term rewrite rule. If $\mathbf{s} \rightarrow \mathbf{t}$ via ρ at position p , then $\mathbf{s}^\circ \rightarrow \mathbf{t}^\circ$ via ρ° at position p .*

Proof. By the definition of a term rewrite step, $\mathbf{s} = C[\mathbf{l}\sigma]$ and $\mathbf{t} = C[\mathbf{r}\sigma]$ for some context C and substitution σ , and $\mathbf{l}\sigma$ is at position p in $C[\mathbf{l}\sigma]$.

By the definitions of encodings and a PBPO⁺ rewrite step, we must show that the diagram

$$\begin{array}{ccccc}
 & & \mathcal{I}(\mathbf{r}) & \xrightarrow{r} & \mathbf{r}^\circ \\
 & & \downarrow \! \! \! \downarrow u & & \downarrow w \\
 \mathbf{I}^\circ & \xrightarrow{m} & (C[\mathbf{l}\sigma])^\circ & \xleftarrow{g_L} & G_K & \xrightarrow{g_R} & (C[\mathbf{r}\sigma])^\circ \\
 \downarrow \text{PB} & & \alpha \downarrow & & \text{PB} \downarrow u' & & \downarrow \text{PO} \\
 \mathbf{I}^\circ & \xrightarrow{t_L} & C[\mathbf{I}^\circ \downarrow \mathcal{X}] & \xleftarrow{l'} & C[\mathcal{I}(\mathbf{r}) \downarrow \mathcal{X}] & &
 \end{array} \tag{1}$$

holds for some G_K and the various morphisms that are not fixed by ρ° (including α), and where m maps $\text{root}(\mathbf{I}^\circ)$ onto position p of $(C[\mathbf{l}\sigma])^\circ$. Note that g_L is a mono by Proposition 26 and stability of monos under pullbacks, and w is a mono by Proposition 15.

By Proposition 30, m and α exist and they exist uniquely. It is then straightforward to check that the middle pullback extracts the subgraphs corresponding to the context C and to every subterm bound to a variable $x \in \text{Var}(\mathbf{l}) \cap \text{Var}(\mathbf{r})$, and that the pushout performs the appropriate gluing around pattern \mathbf{r}° , with $(C[\mathbf{r}\sigma])^\circ$ as the result. \square

Lemma 34 ($(\cdot)^\circ$ Is Closed). *Let $\rho = \mathbf{l} \rightarrow \mathbf{r}$ be a linear term rewrite rule. If $\mathbf{s}^\circ \rightarrow G$ via ρ° then $G \cong \mathbf{t}^\circ$ for some term \mathbf{t} with $\mathbf{s} \rightarrow \mathbf{t}$.*

Proof. Assume $\mathbf{s}^\circ \rightarrow G$ via ρ° at position p . Then by Proposition 32 we have $\mathbf{s}^\circ = (C[\mathbf{l}\sigma])^\circ$ for some context C and substitution σ such that $\mathbf{s}(p) = \square$. Then $\mathbf{s} = C[\mathbf{l}\sigma] \rightarrow C[\mathbf{r}\sigma] = \mathbf{t}$ via ρ at position p . Thus $\mathbf{s}^\circ \rightarrow \mathbf{t}^\circ$ via ρ° at position p by Lemma 33. Then we have $G \cong \mathbf{t}^\circ$ by Lemma 31. \square

Theorem 35. *The encoding $(\cdot)^\circ$ is an embedding.*

Proof. From Lemma 33 and Lemma 34. \square

5 The Embedding Preserves Termination Globally

From the fact that the encoding is step-preserving (Lemma 33), the following is almost immediate.

Lemma 36. *Let R be a linear TRS. If R° is terminating on $\mathbf{FinGraph}^{\Sigma^\circ}$, then R is terminating.* \square

It is obvious that the other direction holds if the category $\mathbf{FinGraph}^{\Sigma^\circ}$ is restricted to graphs that are term encodings; so we have *local termination* [9–11]. However, in this subsection we will show that the direction holds *globally*. Thus, in particular, the finite graphs may be disconnected, cyclic, and labeled arbitrarily from Σ° .

Our overall proof strategy is as follows. First, we show that it suffices to restrict to cycle-free graphs G (Corollary 45). Then, we show that an infinite rewrite sequence on cycle-free G contains (in some sense) an infinite rewrite sequence on term encodings, and therefore on terms (Theorem 62).

Definition 37 (Undirected Path). Let $n \in \mathbb{N}$. An *undirected path* of length n from node v_1 to v_{n+1} in a graph G is a sequence $v_1 e_1 v_2 v_2 \cdots v_n e_n v_{n+1}$ where v_1, v_2, \dots, v_{n+1} are nodes of G and $e_1, e_2, e_3, \dots, e_n$ are edges of G such that $(v_i, v_{i+1}) \in \{(s(e_i), t(e_i)), (t(e_i), s(e_i))\}$ for every $1 \leq i \leq n$.

The path is an *undirected cycle* if moreover $n > 0$, $v_1 = v_{n+1}$ and $e_i \neq e_j$ for all $0 < i < j \leq n$. A *cycle edge* (*cycle node*) is an edge (node) that is part of an undirected cycle. A graph is *cycle-free* if it does not contain undirected cycles.

Example 38. A path of length 1 is an undirected cycle iff its only edge e is a loop, that is, $s(e) = t(e)$. Two edges between two nodes always constitute an undirected cycle of length 2 (irrespective of the direction of the edges).

Proposition 39. *Edge e is a cycle edge iff there exists an undirected path from $s(e)$ to $t(e)$ that does not include e .*

Proof. If $s(e) = t(e)$, one path is the empty path. Obvious otherwise. \square

Proposition 40. *If e is a cycle edge in G and $\phi : G \rightarrow H$ a mono, then $\phi(e)$ is a cycle edge in H .* \square

Although monos preserve the cycle edge property, morphisms do not generally do so (consider a morphism that identifies two parallel edges). However, for adherence morphisms α we have the following result.

Lemma 41. *Consider the PBPO⁺ match square (the leftmost square of the rewrite step diagram) with a host graph G_L . Suppose that e is a cycle edge in G_L and $\alpha(e) = t_L(e')$ for some $e' \in E_L$. Then $\alpha(e)$ is a cycle edge in L' .*

Proof. Let σ_1 be the path just consisting of e . By Proposition 39 there also exists an undirected path from $s(e)$ to $t(e)$ in G_L that does not include e . Since premorphisms preserve undirected paths, $\alpha(\sigma_1)$ and $\alpha(\sigma_2)$ are undirected paths from $\alpha(s(e))$ to $\alpha(t(e))$ in L' . If $\alpha(e)$ is not a cycle edge, then paths $\alpha(\sigma_1)$ and $\alpha(\sigma_2)$ both include $\alpha(e)$ by Proposition 39. Thus α maps two distinct edges in G_L onto $\alpha(e) = t_L(e')$. Since L is the α -preimage of $t_L(L)$, $\alpha \circ m = t_L$ also maps two distinct edges onto $t_L(e)$. This contradicts that t_L is monic. So $\alpha(e)$ is a cycle edge. \square

Lemma 42 (Cycle-Preserving Pullback). *If for $\tau = G \xrightarrow{g} X \xleftarrow{h} H$, (i) σ is an undirected cycle in G , (ii) $g(\sigma)$ lies in the image of h , and (iii) the pullback for τ is $G \xleftarrow{g'} Y \xrightarrow{h'} H$, then every edge $e \in g'^{-1}(\sigma)$ is a cycle edge in Y .* \square

Definition 43 (Cycle Edge Removal). For a graph G , we let $[G]$ denote the graph obtained by deleting all cycle edges from G .

Lemma 44. *Let $\rho : \mathbf{l} \rightarrow \mathbf{r}$ be a linear term rewrite rule over Σ . If there is a rewrite step $G_L \xrightarrow{\rho^\circ} G_R$ on graphs over Σ° , then also $[G_L] \xrightarrow{\rho^\circ} [G_R]$.*

Proof. By the definition of a rewrite step and substituting for the translation of ρ , we have the following arrangement of objects and morphisms

$$\begin{array}{ccccc}
 & & & \mathcal{I}(\mathbf{r}) & \xrightarrow{r} & \mathbf{r}^\circ \\
 & & & \downarrow !u & & \downarrow w \\
 \mathbf{l}^\circ & \xrightarrow{m} & G_L & \xleftarrow{g_L} & G_K & \xrightarrow{g_R} & G_R \\
 \downarrow \text{PB} & & \downarrow \alpha & & \downarrow \text{PB} & & \downarrow \text{PO} \\
 \mathbf{l}^\circ & \xrightarrow{t_L} & \mathcal{C}[\mathbf{l}^\circ \downarrow \mathcal{X}] & \xleftarrow{l'} & \mathcal{C}[\mathcal{I}(\mathbf{r}) \downarrow \mathcal{X}] & &
 \end{array}$$

for some G_K . Many of the morphisms are fixed by the rule ρ° . Note that g_L is a mono by Proposition 26 and stability of monos under pullbacks.

Observe that $t_L(\mathbf{l}^\circ)$ does not contain cycle edges (Definition 25). Hence by Lemma 41, α must map every cycle edge of G_L into one of the edges created by constructing the context closure $\mathcal{C}[\mathbf{l}^\circ \downarrow \mathcal{X}]$ of \mathbf{l}° .

Now suppose that we replace G_L by $[G_L]$ in the diagram. Then the middle pullback object G'_K is obtained by removing from G_K the set of edges $C \subseteq E_{G_K}$ that mono g_L maps into a cycle edge of G_L . Since monos preserve cycle edges, every cycle edge of G_K is in C . Moreover, using Lemma 42, C contains only cycle edges. Hence $G'_K = [G_K]$.

Similarly, the pushout object replacement G'_R for G_R is obtained by removing from G_R the set of edges $C \subseteq E_{G_R}$ that have a cycle edge g_R -preimage in G_K . Since an undirected path ρ in G_R is an undirected cycle iff ρ is in the range of g_R and $g_R^{-1}(\rho)$ is an undirected cycle, $G'_R = [G_R]$. \square

As a direct consequence of Lemma 44 we obtain the following.

Corollary 45. *Let R be a linear TRS over Σ . R° admits an infinite rewrite sequence on all graphs iff R° admits an infinite rewrite sequence on cycle-free graphs.* \square

Thus, in order to prove that termination of R implies termination of R° in $\mathbf{FinGraph}^{\Sigma^\circ}$, it suffices to restrict attention to finite, cycle-free graphs. However, not all such graphs are term-like: graphs may be arbitrarily labeled from Σ° , non-rooted and disconnected. So a further argument is needed.

Definition 46 (Well-Formedness). Let Σ be a signature, and G a graph with labels from Σ° . A node $v \in V_G$ with label $l \in \Sigma \cup \{\perp, \top\} \cup \mathbb{N}^+$ is *in-well-formed* (I) if it has at most one incoming edge; and it is *out-well-formed* (O) if $l \in \Sigma$, and v has precisely $\#l$ outgoing edges, labeled with $1, 2, \dots, \#l$.

Definition 47 (Good and Bad Nodes). A node $v \in V_G$ is called *good* if v is O and all of v 's children in G are I. Nodes that are not good are *bad*.

We will use the distinction between good and bad nodes to define a kind of partitioning on graphs G , which we call a *zoning*. For cycle-free graphs, each zone will be seen to correspond to a term encoding in a qualified sense. (Some edges of G will not be part of any zone of G .) Since most results related to zoning hold not only for non-cycle-free graphs, we will use minimal assumptions where possible (in particular, note that (directed) acyclicity is a weaker condition than cycle-freeness). We do assume finiteness globally.

Definition 48 (Zoning). A *zoning* of G divides up G into *zones*, which are subgraphs of G . The zoning is iteratively constructed as follows:

- Initially, every node of G forms its own zone.
- At each subsequent iteration, if an edge e is not included in a zone and $s(e)$ is good, join the zones of $s(e)$ and $t(e)$ along e . (If $s(e)$ and $t(e)$ are in the same zone Z , this is the same as adding e to zone Z .)
- The algorithm terminates if the previous step can no longer be applied.

Definition 49 (Bridge). A *bridge* is an edge $e \in E_G$ not included in any zone of G .

Proposition 50. *The zoning of a graph G is unique, and any zone is a connected subgraph.* □

Proposition 51. *If $e \in E_Z$ is included in zone Z , then $t(e)$ is I.*

Proof. Since e was joined along, $s(e)$ is good, and hence $t(e)$ is I. □

Definition 52 (Root). A node $v \in V_Z$ without a parent inside zone Z is called a *root* for Z .

Proposition 53. *Within a zone Z , for any two nodes $u, v \in V_Z$, there is a node $x \in V_Z$ such that $u \leftarrow^* x \rightarrow^* v$ (using edges included in Z).*

Proof. Because any zone is connected, there is an undirected path between u and v within Z . This path cannot contain a segment of the form $a \rightarrow c \leftarrow b$, for then c would not be I, contradicting Proposition 51. Hence the path must be of the form $u \leftarrow^* x \rightarrow^* v$ for some $x \in V_Z$. □

Corollary 54. *If a zone has a root, it is unique.*

Proposition 55. *If a zone Z is acyclic, it has a root.*

Proof. If not, following the edges in Z backwards would reveal a directed cycle in Z . □

Proposition 56. *If a zone Z is acyclic, then Z is a directed tree.*

Proof. As follows from the preceding propositions, Z is connected and each zone has a unique root u .

By Proposition 53, u has a path to every node v in Z . Such an (acyclic) path is moreover unique, for otherwise the first point at which these paths join is not I, contradicting Proposition 51. Thus Z is a directed tree. □

We also have the following general characterization of bridges.

Proposition 57 (On Bridges). *The source of a bridge is a bad leaf of a zone, and the target of a bridge is a root of a zone.*

Proof. If e is a bridge, $s(e)$ must be bad. If $s(e)$ is bad, none of its outgoing edges have been joined along. Hence $s(e)$ is a leaf in Z .

If a bridge e targets a non-root $t(e)$ of a zone Z , then $t(e)$ is not I, since it has at least two incoming edges. Thus the parent p of $t(e)$ inside Z is bad. But this contradicts that p must be good since it has an edge to $t(e)$ inside Z . Hence $t(e)$ must be a root. \square

Although acyclic zones are directed trees, not every zone corresponds directly to a term encoding \mathbf{t}° for some term \mathbf{t} . For instance, for the 3-zone graph $f \xrightarrow{1} a \xleftarrow{1} f$, with $\#(f) = 1$ and $\#(a) = 0$, only the zone containing the node labeled with a corresponds to a term encoding. But we have the following result.

Proposition 58 (Zones as Term Encodings). *If every bad node of an acyclic zone Z is relabeled with \perp , then Z is isomorphic to a term encoding \mathbf{t}° .*

Proof. Every acyclic zone is structurally a directed tree. All inner nodes (and some leaves labeled with constants $a \in \Sigma$) are good, meaning they are labeled with Σ and out-well-formed; and all of their children are in-well formed and included into the zone by the zoning algorithm. Since bad nodes are leaves, relabeling them with \perp essentially makes them represent variables. To establish an isomorphism between a zone and a term encoding, one simply has to rename the identity of every good node to its position in this tree, and the identity of every bad node to some unique $x \in \mathcal{X}$. \square

We will now show that relabeling bad nodes with \perp does not meaningfully affect the rewriting behavior in a graph G . Intuitively, this is because matches cannot cross zones, as shown by the following results. Recall the terminology of Definition 23.

Lemma 59. *A match morphism $m : \mathbf{I}^\circ \rightarrow G$ (for a rule encoding ρ°) maps symbol vertices $v \in V_{\rho^\circ}$ onto good nodes.*

Proof. We must show that $m(v)$ is O and that all of $m(v)$'s children are I.

First, we show that $m(v)$ is O. Because v is a symbol vertex, $\ell(v) \in \Sigma$. Since morphisms do not decrease labels, either (a) $\ell(v) = \ell(m(v))$, or (b) $\ell(v) < \ell(m(v))$.

In case (a), we must show that $m(v)$ has precisely $\#(\ell(m(v))) = \#(\ell(v))$ outgoing edges labeled with $1, 2, \dots, \#(\ell(v))$. By monicity of m and the definition of rule encodings, we know that it has these edges at least once. Moreover, $m(v)$ cannot have additional outgoing edges, since these cannot be suitably mapped by α into L' without violating the strong match property.

In case (b), we obtain a contradiction. For note that $t_L : \mathbf{I}^\circ \rightarrow \mathcal{C}[\mathbf{I}^\circ \downarrow \mathcal{X}]$ preserves labels for nodes labeled from Σ , so that $\ell(t_L(v)) = \ell(v)$. Furthermore, since m enables a rewrite step, $t_L = \alpha \circ m$ and hence $\ell((\alpha \circ m)(v)) = \ell(v)$. This implies that α decreases the label on $m(v)$, which is not allowed by the \leq requirement on morphisms.

Second, we establish that *all of $m(v)$'s children are I*. Observe that for symbol vertices v , all incoming edges of children of $t_L(v)$ (i) have their source in $t_L(v)$ and (ii) are in the image of t_L . For a contradiction, assume a child u of $m(v)$ has multiple incoming edges e, e' . Then using that $\alpha(m(v)) = t_L(v)$ (by the strong match property) and that $\alpha(u)$ is a child of $t_L(v)$, by observation (i) $\alpha(s(e)) = \alpha(s(e'))$. Since there are no parallel edges in L' , $\alpha(e) = \alpha(e')$. By (ii) $\alpha(e)$ is in the image of t_L . Thus multiple elements are mapped onto the same element in L' . This violates the strong match property. Contradiction. \square

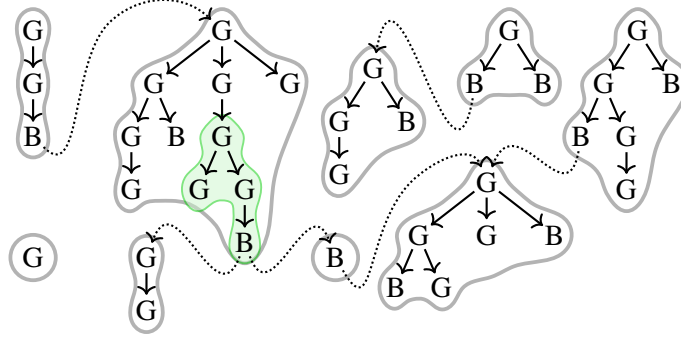


Figure 1: A zoning of a cycle-free graph with three components. Zone borders (gray), good (G) and bad (B) nodes, bridges (dotted) and a match (green) are indicated.

Lemma 60 (Matches Respect Boundaries). *Let $\rho = \mathbf{l} \rightarrow \mathbf{r}$ be a TRS rule, and consider the translation ρ° . Then for any match morphism $m : \mathbf{l}^\circ \rightarrow G$, the image $m(\mathbf{l}^\circ)$ lies in precisely one zone.*

Proof. Because \mathbf{l}° is connected, so is $m(\mathbf{l}^\circ)$. So if a counterexample to the lemma exists, it involves a bridge. Let $m(e)$, the image of an $e \in E_{\mathbf{l}^\circ}$, be such a bridge. By Proposition 57, $s(m(e)) = m(s(e))$ is a bad leaf of a zone Z . Hence $s(e) \in V_{\mathbf{l}^\circ}$ is a variable head by the contrapositive of Proposition 59. Since variable heads are leaves in \mathbf{l}° , this contradicts that $s(e)$ has $e \in E_{\mathbf{l}^\circ}$ for an outgoing edge. \square

Figure 1 is an abstract depiction of a zoning, and exemplifies the properties established thus far.

Proposition 61 (Bad Node Labels Are Irrelevant). *Let $G^{[\ell(v):=l]}$ denote the graph obtained by changing the label of $v \in V_G$ to $l \in \mathcal{L}$. If $v \notin V_G$, $G^{[\ell(v):=l]} = G$.*

For bad $v \in V_G$ and any $l \in \mathcal{L}$, if $G \rightarrow H$ is a rewrite step via a translated TRS rule ρ° and adherence morphism α , then $G^{[\ell(v):=l]} \rightarrow H^{[\ell(v):=l]}$ is a rewrite step via ρ° and α .

Proof. In a rewrite step, bad nodes are either matched by variable heads, or lie outside the image of t_L . In both cases, the label does not influence the application condition, since any label l with $\perp \leq l \leq \top$ is allowed. Moreover, the node is either preserved (and its label unchanged), or deleted. In either case the statement holds. \square

Theorem 62. *Let R be a linear TRS. R is terminating on $\text{Ter}(\Sigma, \mathcal{X})$ iff R° is terminating on $\mathbf{FinGraph}^{\Sigma^\circ}$.*

Proof. Direction \Leftarrow is Lemma 36.

For direction \Rightarrow , we prove the contrapositive. By Corollary 45, we may assume G is cycle-free, and thus acyclic. So suppose R° admits an infinite rewrite sequence $\tau_G = G \rightarrow G' \rightarrow \dots$ rooted in a cycle-free, finite graph G .

Because matches respect zone boundaries, the number of zones is finite, and zones are never created by rewrite steps, there exists a zone Z of G in which a match is fixed and rewritten infinitely often. This zone is at no point affected by matches in other zones, since zones can only affect other zones by completely deleting them. Similarly, due to cycle-freeness, it is easy to see that the bridges and zones connected to Z do not affect rule applicability in Z . Hence we can restrict G to Z , and construct an infinite rewrite sequence $\tau_Z = Z \rightarrow Z' \rightarrow Z'' \rightarrow \dots$.

By relabeling every bad node of starting term Z with \perp , the existence of an infinite rewrite sequence is not disturbed using Proposition 61. Furthermore, Z is now isomorphic to a term encoding \mathbf{t}° for some term \mathbf{t} (Proposition 58). Using the fact that the encoding is closed (Lemma 34) and that rewriting is defined modulo isomorphism, we can obtain an infinite rewrite sequence on terms. Thus R is also not terminating. \square

Remark 63. Our result may be compared to one due to Nolte [15, Chapter 6]. Nolte first defines two encodings of TRSs into term graph rewriting systems, a basic encoding and an extended encoding. These encodings preserve neither termination nor confluence, and are not embeddings. He then shows that for term graph systems obtained by the basic encoding, there exists a globally termination-preserving encoding into graph rewriting systems (DPO) [15, Theorem 6.3]. So although Nolte’s approach is similar to ours in spirit, it does not constitute a globally termination-preserving embedding of TRSs into graph rewriting systems.

Remark 64 (Confluence). Although \Leftarrow of Theorem 62 holds for confluence as well, \Rightarrow does not, even if graphs are assumed to be connected, cycle-free and well-labeled. Namely, consider $\Sigma = \{f, g, h, a, b\}$ with $\#(f) = \#(g) = \#(h) = 1$ and $\#(a) = \#(b) = 0$, and the confluent TRS $R = \{g(x) \rightarrow a, h(x) \rightarrow b\}$. Then for the graph $g \xrightarrow{1} f \xrightarrow{1} a \xleftarrow{1} f \xleftarrow{1} h$ both a and b are R° -normal forms.

If graphs may be disconnected, rule $g(x) \rightarrow a$ even constitutes a counter-example by itself. For the type graph of its rule encoding, a disjoint component H can either be mapped onto the upper context closure (preserving H) or the lower context closure (deleting H).

6 Discussion

We have defined an encoding of linear term rewriting into PBPO^+ rewriting that is both an embedding and globally termination-preserving. These properties are achievable because a PBPO^+ rule allows (i) specifying where parts of a context may occur around a pattern, (ii) ensuring that these parts are disjoint, and (iii) deleting such parts (in our case study, such parts correspond to variable substitutions).

We submit that a rewriting framework \mathcal{F} can be said to be a proper generalization of some other framework \mathcal{G} if there exists an embedding \mathcal{E} from \mathcal{G} to \mathcal{F} . In this sense, PBPO^+ is a proper generalization of linear term rewriting (and DPO is not). Often we want the encoding \mathcal{E} to have additional properties such as the global preservation of certain properties (e.g., termination). For instance, the embedding that interprets the TRS rule $\rho = a(b(x)) \rightarrow b(a(x))$ as a mere swap of symbols, and thus as applicable in any context, is an embedding that does not preserve termination globally. (Note that such an alternative embedding is also expressible in PBPO^+ .)

The fact that a certain property-preserving embedding is possible is an interesting expressiveness result for the embedding formalism. Moreover, it opens up a path to reduction arguments, as was also considered by Nolte [15] in a different setting (Remark 63). In our case, if a PBPO^+ rewrite system is (isomorphic to) the encoding of a TRS (as defined in Definition 25), termination can be decided by considering the decoded TRS and forgetting about the complexities of graphs. Our proof technique extends to more general PBPO^+ rewrite systems as long as the following conditions are met: the pattern of the rules is tree-like (possibly with loops on the nodes of the pattern), the outgoing edges of nodes in the pattern have distinct labels, and the ‘context’ and ‘variable graphs’ are disconnected (except through the pattern) and are not duplicated by the rule.

Our provided embedding into PBPO^+ does not preserve confluence globally. As shown in Remark 64, the key problem is that an assumption true for terms, namely connectedness, does not hold

for graphs. For the same reason it is currently impossible to define a termination-preserving embedding of non-right-linear term rewriting into $PBPO^+$: whenever a variable is duplicated, it may also lead to the duplication of any number of disjoint components in the graph that are mapped onto the corresponding variable closure. For future work, we intend to investigate extensions of our encoding that do preserve confluence and termination globally even when variables are duplicated.

Adopting a broader perspective, we hope that our encoding contributes to the development of termination techniques for graph rewriting. There have been recent advances in proving termination of graph rewriting; see for instance work by Bruggink et al. [2] and Dershowitz et al. [6]. In [6], recursive path orders are generalized from term rewriting to graph transformation by decomposing the graph into strongly connected components and a well-founded structure between them. A difficulty in this approach is that all possible cycles around the pattern of a rule must be considered. We hope that the technique can be extended to $PBPO^+$ and strengthened by making use of the application conditions that exclude certain cycles around and through the pattern.

Finally, we believe that our result is a step towards modeling other rewriting formalisms such as lambda calculus and higher order rewriting using graph rewriting. These formalisms also rewrite tree structures, and we expect that extensions of our zoning construction will be instrumental for this purpose. Our goal in this respect is to model these systems in such a way that important properties like termination and confluence are preserved globally, while at the same time keeping the modeling overhead minimal (e.g., avoiding auxiliary rules and rewrite steps that increase the length of rewrite sequences).

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