

On the Strong Equivalences of LP^{MLN} Programs

Bin Wang Jun Shen Shutao Zhang Zhizheng Zhang

School of Computer Science and Engineering
Southeast University, Nanjing, China

{kse.wang, junshen, shutao_zhang, seu_zzz}@seu.edu.cn

By incorporating the methods of Answer Set Programming (ASP) and Markov Logic Networks (MLN), LP^{MLN} becomes a powerful tool for non-monotonic, inconsistent and uncertain knowledge representation and reasoning. To facilitate the applications and extend the understandings of LP^{MLN} , we investigate the strong equivalences between LP^{MLN} programs in this paper, which is regarded as an important property in the field of logic programming. In the field of ASP, two programs P and Q are strongly equivalent, iff for any ASP program R , the programs $P \cup R$ and $Q \cup R$ have the same stable models. In other words, an ASP program can be replaced by one of its strong equivalent without considering its context, which helps us to simplify logic programs, enhance inference engines, construct human-friendly knowledge bases etc. Since LP^{MLN} is a combination of ASP and MLN, the notions of strong equivalences in LP^{MLN} is quite different from that in ASP. Firstly, we present the notions of p-strong and w-strong equivalences between LP^{MLN} programs. Secondly, we present a characterization of the notions by generalizing the SE-model approach in ASP. Finally, we show the use of strong equivalences in simplifying LP^{MLN} programs, and present a sufficient and necessary syntactic condition that guarantees the strong equivalence between a single LP^{MLN} rule and the empty program.

1 Introduction

LP^{MLN} [9], a newly developed knowledge representation and reasoning language, is designed to handle non-monotonic and uncertain knowledge by combining the methods of Answer Set Programming (ASP) [2, 6] and Markov Logic Networks (MLN) [15]. Specifically, an LP^{MLN} program can be viewed as a weighted ASP program, where each ASP rule is assigned a weight denoting its certainty degree, and each weighted rule is allowed to be violated by a set of beliefs associated with the program. For example, LP^{MLN} rule “ $w : \leftarrow a, b.$ ” is a weighted constraint denoting facts a and b are contrary, w is the weight of the constraint. In the view of ASP, the set $X = \{a, b\}$ is impossible to be a belief set of any ASP programs containing the constraint, while in the context of LP^{MLN} , X is a valid belief set. Since X violates the constraint, the weight $-w$ is regarded as the certainty degree of X . It is easy to observe that the example can also be encoded by weak constraints in ASP. From this perspective, LP^{MLN} can be viewed as an extension of ASP with weak constraints, that is, ASP with weak rules. Besides, several inference tasks are introduced to LP^{MLN} such as computing marginal probability distribution of beliefs, computing most probable belief sets etc., which makes LP^{MLN} suitable for knowledge reasoning in the context that contains uncertain and inconsistent data. For example, Eiter and Kaminski [5] used LP^{MLN} in the tasks of classifying visual objects, and some unpublished work tried to use LP^{MLN} as the bridge between text and logical knowledge bases.

Recent results on LP^{MLN} aim to establish the relationships among LP^{MLN} and other logic formalisms [1, 11], develop LP^{MLN} solvers [8, 17, 19], acquire the weights of rules automatically [10], explore the properties of LP^{MLN} [18] etc. All these results lay the foundation for the problems solving via LP^{MLN} , however, many theoretical problems of LP^{MLN} are still unsolved, which prevents the wider

applications of LP^{MLN} . In this paper, we investigate the strong equivalences between LP^{MLN} programs, which is regarded as an important property in the field of logic programming. For two ASP programs P and Q , they are strongly equivalent, iff for any ASP program R , the programs $P \cup R$ and $Q \cup R$ have the same stable models [12]. In other words, an ASP program can be replaced by one of its strong equivalent without considering its context, which helps us to simplify logic programs, enhance inference engines, construct human-friendly knowledge bases etc. For example, an ASP rule such that its positive and negative body have common atoms is strongly equivalent to \emptyset [7, 13, 14], therefore, such kinds of rules can be eliminated from any context, which leads to a more concise knowledge base and makes the reasoning easier. By investigating the strong equivalences in LP^{MLN} , it is expected to improve the knowledge base constructing and knowledge reasoning in LP^{MLN} , furthermore, help us to facilitate the applications and extend the understandings of LP^{MLN} .

Our contributions are as follows. Firstly, we define the notions of strong equivalences in LP^{MLN} , that is, the p-strong and w-strong equivalences. As we showed in above example, a stable model defined in LP^{MLN} is associated with a certainty degree, therefore, the notions of strong equivalences in LP^{MLN} are also relevant to the certainty degree. Secondly, we present a model-theoretical approach to characterizing the defined notions, which can be viewed as a generalization of the strong-equivalence models (SE-model) approach in ASP [16]. Finally, we show the use of the strong equivalences in simplifying LP^{MLN} programs, and present a sufficient and necessary syntactic condition that guarantees the strong equivalences between a single LP^{MLN} rule and the empty program.

2 Preliminaries

In this section, we review the knowledge representation and reasoning language LP^{MLN} presented in [9]. An LP^{MLN} program is a finite set of weighted rules $w : r$, where w is the weight of rule r , and r is an ASP rule of the form

$$l_1 \vee \dots \vee l_k \leftarrow l_{k+1}, \dots, l_m, \text{ not } l_{m+1}, \dots, \text{ not } l_n. \quad (1)$$

where l_s are literals, \vee is epistemic disjunction, and *not* is default negation. The weight w of an LP^{MLN} rule is either a real number or a symbol “ α ” denoting “infinite weight”, and if w is a real number, the rule is called *soft*, otherwise, it is called *hard*. For convenient description, we introduce some notations. By \bar{M} we denote the set of unweighted ASP counterpart of an LP^{MLN} program M , i.e. $\bar{M} = \{r \mid w : r \in M\}$. For an ASP rule r of the form (1), the literals occurred in head, positive body, and negative body of r are denoted by $h(r) = \{l_i \mid 1 \leq i \leq k\}$, $b^+(r) = \{l_i \mid k+1 \leq i \leq m\}$, and $b^-(r) = \{l_i \mid m+1 \leq i \leq n\}$ respectively. Therefore, an ASP rule r of the form (1) can also be abbreviated as “ $h(r) \leftarrow b^+(r), \text{ not } b^-(r)$ ”. By $lit(r) = h(r) \cup b^+(r) \cup b^-(r)$ we denote the set of literals occurred in rule r , and by $lit(\Pi) = \bigcup_{r \in \Pi} lit(r)$ we denote the set of literals occurred in an ASP program Π .

An LP^{MLN} program is called *ground* if its rules contain no variables. Usually, a non-ground LP^{MLN} program is considered as a shorthand for the corresponding ground program, therefore, we limited our attention to the strong equivalences between ground LP^{MLN} programs in this paper. For a ground LP^{MLN} program M , we use $W(M)$ to denote the weight degree of M , i.e. $W(M) = \exp(\sum_{w:r \in M} w)$. A ground LP^{MLN} rule $w : r$ is satisfied by a consistent set X of ground literals, denoted by $X \models w : r$, if $X \models r$ by the notion of satisfiability in ASP. An LP^{MLN} program M is satisfied by X , denoted by $X \models M$, if X satisfies all rules in M . By M_X we denote the LP^{MLN} *reduct* of an LP^{MLN} program M w.r.t. X , i.e. $M_X = \{w : r \in M \mid X \models w : r\}$. A consistent set X of literals is a stable model of an ASP program P , if X satisfies all rules in P^X and X is minimal in the sense of set inclusion, where P^X is the Gelfond-Lifschitz reduct (GL-reduct) of P w.r.t. X , i.e. $P^X = \{h(r) \leftarrow b^+(r). \mid r \in P \text{ and } b^-(r) \cap X = \emptyset\}$. The set X is a

stable model of an LP^{MLN} program M if X is a stable model of the ASP program $\overline{M_X}$. And by $SM(M)$ we denote the set of all stable models of an LP^{MLN} program M . For a stable model X of an LP^{MLN} program M , the *weight degree* $W(M, X)$ of X w.r.t. M is defined as $W(M_X)$, and the *probability degree* $P(M, X)$ of X w.r.t. M is defined as

$$P(M, X) = \lim_{\alpha \rightarrow \infty} \frac{W(M, X)}{\sum_{X' \in SM(M)} W(M, X')} \quad (2)$$

For a literal l , the *probability degree* $P(M, l)$ of l w.r.t. M is defined as

$$P(M, l) = \sum_{l \in X, X \in SM(M)} P(M, X) \quad (3)$$

A stable model X of an LP^{MLN} program M is called a *probabilistic stable model* of M if $P(M, X) \neq 0$. By $PSM(M)$ we denote the set of all probabilistic stable models of M . It is easy to check that X is a probabilistic stable model of M , iff X is stable model of M that satisfies the most hard rules. Based on above definitions, there are two kinds of main inference tasks for an LP^{MLN} program M [8]:

- Maximum A Posteriori (MAP) inference: compute the stable models with the highest weight or probability degree of the program M , i.e. the most probable stable model;
- Marginal Probability Distribution (MPD) inference: compute the probability degrees of a set of literals w.r.t. the program M .

3 Strong Equivalences for LP^{MLN}

In this section, we investigate the strong equivalences in LP^{MLN}. Firstly, we define the notions of strong equivalences based on two different certainty degrees in LP^{MLN}. Secondly, we present a model-theoretical approach to characterizing the notions. Finally, we present the relationships among these notions.

3.1 Notions of Strong Equivalences

The notion of strong equivalence is built on the notion of ordinary equivalence, in this section, we define two notions of ordinary equivalences between LP^{MLN} programs, which is relevant to the weight and probability defined for stable models in LP^{MLN}.

Definition 1 (w-ordinary equivalence). *Two LP^{MLN} programs L and M are w-ordinarily equivalent, denoted by $L \equiv_w M$, if their stable models coincide, and for each stable model X of the programs, $W(L, X) = W(M, X)$.*

Definition 2 (p-ordinary equivalence). *Two LP^{MLN} programs L and M are p-ordinarily equivalent, denoted by $L \equiv_p M$, if their stable models coincide, and for each stable model X of the programs, $P(L, X) = P(M, X)$.*

From Definition 1 and Definition 2, it can be observed that both of the w-ordinary and p-ordinary equivalences can guarantee two LP^{MLN} programs have the same MAP and MPD inference results, and the p-ordinary equivalence is a little weaker, i.e. if two LP^{MLN} programs are p-ordinarily equivalent, then they are w-ordinarily equivalent, but the inverse does not hold generally. Based on the definitions of ordinary equivalences, we can define two kinds of strong equivalences between LP^{MLN} programs.

Definition 3 (strong equivalences for LP^{MLN}). *For two LP^{MLN} programs L and M ,*

- they are w -strongly equivalent, denoted by $L \equiv_{s,w} M$, if for any LP^{MLN} program N , $L \cup N \equiv_w M \cup N$;
- they are p -strongly equivalent, denoted by $L \equiv_{s,p} M$, if for any LP^{MLN} program N , $L \cup N \equiv_p M \cup N$.

The notions of w -strong and p -strong equivalences can guarantee the faithful replacement of an LP^{MLN} program in any context. Here, we introduce a new notion of strong equivalence, semi-strong equivalence, that does not guarantee the faithful replacement, but helps us to simplify the characterizations of other strong equivalences.

Definition 4 (semi-strong equivalence). *Two LP^{MLN} programs L and M are semi-strongly equivalent, denoted by $L \equiv_{s,s} M$, if for any LP^{MLN} program N , the programs $L \cup N$ and $M \cup N$ have the same stable models.*

3.2 Characterizations of Strong Equivalences

In this section, we present the characterizations for w -strong and p -strong equivalences. From Definition 3 and Definition 4, the notions of w -strong and p -strong equivalences can be viewed as the strengthened semi-strong equivalence by introducing the certainty evaluations. Therefore, we present the characterization of semi-strong equivalence firstly, which serves as the basis of characterizing w -strong and p -strong equivalences.

3.2.1 Characterizing Semi-Strong Equivalence

Here, we characterize the semi-strong equivalence between LP^{MLN} programs by generalizing the strong-equivalence models (SE-models) approach presented in [16]. For the convenient description, we introduce following notions.

Definition 5 (SE-interpretation). *A strong equivalence interpretation (SE-interpretation) is a pair of consistent sets of literals (X, Y) such that $X \subseteq Y$. An SE-interpretation (X, Y) is called total if $X = Y$, and non-total if $X \subset Y$.*

Definition 6 (SE-models for LP^{MLN}). *For an LP^{MLN} program M , an SE-interpretation (X, Y) is an SE-model of M , if $X \models M'$ and $Y \models M'$, where $M' = (\overline{M_Y})^Y$.*

In Definition 6, M' is an ASP program obtained from M by a three-step transformation. In the first step, M_Y is obtained from M by removing all rules that cannot be satisfied by Y , which is the LP^{MLN} reduct of M w.r.t. Y . In the second step, $\overline{M_Y}$ is obtained by dropping weight of each rule in M_Y . In the third step, $(\overline{M_Y})^Y$ is obtained by the GL-reduct. Clearly, an SE-model for the LP^{MLN} program M is an SE-model of a consistent unweighted subset of M that is obtained by LP^{MLN} reduct, which means the definition of SE-models for LP^{MLN} programs is built on the definition of SE-models for ASP programs. In what follows, we use $LSE(M)$ to denote the set of all SE-models of an LP^{MLN} program M .

Definition 7. *For an LP^{MLN} program M and an SE-model (X, Y) of M , the weight degree $W(M, (X, Y))$ of (X, Y) w.r.t. the program M is defined as*

$$W(M, (X, Y)) = W(M_Y) = \exp \left(\sum_{w:r \in M_Y} w \right) \quad (4)$$

Example 1. *Consider an LP^{MLN} program $L = \{\alpha : a \vee b. \quad 1 : b \leftarrow \text{not } a.\}$. For the set $X = \{a, b\}$, it is easy to check that $X \models L$, therefore, the LP^{MLN} reduct L_X is L itself. By the definitions of GL-reduct, $(\overline{L})^X = \{a \vee b.\}$, therefore, both $S_1 = (\{a\}, X)$ and $S_2 = (\{b\}, X)$ are SE-models of L , and $W(L, S_1) = W(L, S_2) = W(L) = e^{\alpha+1}$.*

Now, we show some useful properties of the SE-models for LP^{MLN} programs. Proposition 1 is an immediate result according to the definition of SE-models.

Proposition 1. *Let M be an LP^{MLN} program and (X, Y) an SE-interpretation,*

- *if $X = Y$, then (X, Y) is an SE-model of M ;*
- *(X, Y) is not an SE-model of M , iff $X \not\models (\overline{M_Y})^Y$.*

Proposition 2 shows the relationships between the SE-models and the stable models of an LP^{MLN} program.

Proposition 2. *For an LP^{MLN} program M and a total SE-model (X, X) of M ,*

- *there must be an LP^{MLN} program N such that X is a stable model of $M \cup N$, for example, $N = \{w : a. \mid a \in X\}$;*
- *X is a stable model of M , iff $(X', X) \notin LSE(M)$ for any proper subset X' of X .*

Based on above results, a characterization of semi-strong equivalence between LP^{MLN} programs is presented in Lemma 1.

Lemma 1. *Let L and M be two LP^{MLN} programs, they are semi-strongly equivalent, iff they have the same SE-models, i.e. $LSE(L) = LSE(M)$.*

Proof. The proof proceeds basically along the lines of the corresponding proof by Turner [16].

For the if direction, suppose $LSE(L) = LSE(M)$, we need to prove that for any LP^{MLN} program N , the programs $L \cup N$ and $M \cup N$ have the same stable models. We use proof by contradiction. Assume Y is a set of literals such that $Y \in SM(L \cup N) - SM(M \cup N)$. By the definition, we have $Y \models (\overline{(L \cup N)_Y})^Y = (\overline{L_Y})^Y \cup (\overline{N_Y})^Y$. By Proposition 1, we have (Y, Y) is an SE-model of L . Hence, (Y, Y) is also an SE-model of M . Then, we have $Y \models (\overline{M_Y})^Y$ and $Y \models (\overline{(M \cup N)_Y})^Y$. By the assumption $Y \notin SM(M \cup N)$, there exists a consistent set X of literals such that $X \models (\overline{(M \cup N)_Y})^Y$, then we have $X \models (\overline{M_Y})^Y$ and $X \models (\overline{N_Y})^Y$, hence, (X, Y) is an SE-model of M , which means (X, Y) is also an SE-model of L . By the definition of stable model, Y cannot be a stable model of $L \cup N$, which contradicts with the assumption $Y \in SM(L \cup N)$. Therefore, the programs $L \cup N$ and $M \cup N$ have the same stable models, and the if direction of Lemma 1 is proven.

For the only-if direction, suppose $SM(L \cup N) = SM(M \cup N)$, we need to prove that $LSE(L) = LSE(M)$. We use proof by contradiction. Assume (X, Y) is an SE-interpretation such that $(X, Y) \in LSE(L) - LSE(M)$. By Proposition 1, we have $X \not\models (\overline{M_Y})^Y$. Let $N = \{1 : a. \mid a \in X\} \cup \{1 : a \leftarrow b. \mid a, b \in Y - X\}$. We have $(\overline{(M \cup N)_Y})^Y = (\overline{M_Y})^Y \cup \overline{N}$. Let X' be a set of literals such that $X' \subseteq Y$ and $X' \models (\overline{M_Y})^Y \cup \overline{N}$. By the construction of N , we have $X \subseteq X'$. Since $X \not\models (\overline{M_Y})^Y$, we have $X \neq X'$. Hence, there must exist a literal $l \in Y - X$ such that $l \in X'$. By the construction of N , we have $(Y - X) \subseteq X'$, which means $X' = Y$. By the definition of stable models, Y is a stable model of $M \cup N$, which means Y should also be a stable model of $L \cup N$. By the definition of stable model, (X, Y) cannot be an SE-model of L , which contradicts with the assumption $(X, Y) \in LSE(L)$. Therefore, L and M have the same SE-models, and the only-if direction of Lemma 1 is proven. \square

3.2.2 Characterizing W-Strong and P-Strong Equivalences

Now we present a main result of the paper, that is, the characterizations of w-strong and p-strong equivalences. Based on Lemma 1, Lemma 2 provides a sufficient condition to characterize the p-strong equivalence for LP^{MLN} programs.

Lemma 2. *Two LP^{MLN} programs L and M are p-strongly equivalent, if $LSE(L) = LSE(M)$, and there exist two constants c and k such that for each SE-model $(X, Y) \in LSE(L)$, $W(L, (X, Y)) = \exp(c + k * \alpha) * W(M, (X, Y))$.*

Proof. For two LP^{MLN} programs L and M , by Lemma 1, if $LSE(L) = LSE(M)$, L and M are semi-strongly equivalent, i.e. for any LP^{MLN} program N , $SM(L \cup N) = SM(M \cup N)$. Suppose there exist two constants c and k such that for each SE-model $(X, Y) \in LSE(L)$, $W(L, (X, Y)) = \exp(c + k * \alpha) * W(M, (X, Y))$, we need to show that L and M are p-strongly equivalent. Let N be an LP^{MLN} program, it is easy to check that X is a probabilistic stable model of $L \cup N$ iff X is a probabilistic stable model of $M \cup N$, i.e. $PSM(L \cup N) = PSM(M \cup N)$. For a stable model $X \in PSM(L \cup N)$, the probability degree of X can be reformulated as

$$\begin{aligned} P(L \cup N, X) &= \frac{W(L \cup N, X)}{\sum_{X' \in PSM(L \cup N)} W(L \cup N, X')} = \frac{\exp(c + k * \alpha) * W(M \cup N, X)}{\exp(c + k * \alpha) * \sum_{X' \in PSM(M \cup N)} W(M \cup N, X')} \\ &= \frac{W(M \cup N, X)}{\sum_{X' \in PSM(M \cup N)} W(M \cup N, X')} = P(M \cup N, X) \end{aligned} \quad (5)$$

By the definition of p-strong equivalence, we have $L \equiv_{s,p} M$. \square

The condition in Lemma 2, called PSE-condition, is sufficient to characterize the p-strong equivalence. One may ask that whether the PSE-condition is also necessary. To answer the question, we need to consider the hard rules of LP^{MLN} particularly. For LP^{MLN} programs containing no hard rules, it is easy to check that the PSE-condition is necessary. But for arbitrary LP^{MLN} programs, this is not an immediate result, which is shown as follows. Firstly, we introduce some notations. For a set U of literals, we use 2^U to denote the power set of U , and use 2^{U^+} to denote the maximal consistent part of the power set of U , i.e. $2^{U^+} = \{X \in 2^U \mid X \text{ is consistent}\}$.

Lemma 3. *For two p-strongly equivalent LP^{MLN} programs L and M , let N_1 and N_2 be arbitrary LP^{MLN} programs such that $PSM(L \cup N_1) \cap PSM(L \cup N_2) \neq \emptyset$. There exist two constants c and k such that for any SE-models (X, Y) of L , if $Y \in PSM(L \cup N_1) \cup PSM(L \cup N_2)$, then $W(L, (X, Y)) = \exp(c + k * \alpha) * W(M, (X, Y))$.*

By Lemma 3, for two p-strongly equivalent LP^{MLN} programs L and M , to prove the necessity of the PSE-condition, we need to find a set E of LP^{MLN} programs satisfying

- $\forall N_1, N_2 \in E, PSM(L \cup N_1) \cap PSM(L \cup N_2) \neq \emptyset$; and
- $\bigcup_{N \in E} PSM(L \cup N) = 2^{U^+}$, where U is the set of literals occurred in L and M , i.e. $U = \text{lit}(\overline{L \cup M})$.

Above set E is called a set of necessary extensions w.r.t. LP^{MLN} programs L and M . As shown in Proposition 1, an arbitrary total SE-interpretation is an SE-model of an LP^{MLN} program, therefore, if there exists a set of necessary extensions of two p-strongly equivalent programs L and M , then the necessity of the PSE-condition can be proven. In what follows, we present a method to construct a set of necessary extensions.

Definition 8. For two consistent sets X and Y of literals, and an atom a such that $a \notin X \cup Y$, by $R(X, Y, a)$ we denote an LP^{MLN} program as follows

$$\alpha : \leftarrow X, \text{ not } Y, a. \quad (6)$$

$$\alpha : a \leftarrow X, \text{ not } Y. \quad (7)$$

Definition 9 (flattening extension). For an LP^{MLN} program M and a set U of literals such that $\text{lit}(\overline{M}) \subseteq U$, a flattening extension $T^k(M, U)$ of M w.r.t. U is defined as

- $T^0(M, U) = M \cup N_0$;
- $T^{i+1}(M, U) = T^i(M, U) \cup R(X \cap U, U - X, a_{i+1})$,

where N_0 is a set of weighted facts constructed from U , i.e. $N_0 = \{\alpha : a_k. \mid a_k \in U\}$, X is a probabilistic stable model of $T^i(M, U)$, i.e. $X \in \text{PSM}(T^i(M, U))$, and $a_{i+1} \notin \text{lit}(\overline{T^i(M, U)})$.

According to the splitting set theorem of LP^{MLN} [18], the flattening extension has following properties.

Proposition 3. For an LP^{MLN} program M and a set U of literals, if $T^{k+1}(M, U)$ is constructed from $T^k(M, U)$ by adding $R(X \cap U, U - X, a_{k+1})$, then we have

- $SM(T^0(M, U)) = 2^{U^+}$;
- $SM(T^{k+1}(M, U)) = SM(T^k(M, U)) \cup \{Y \cup \{a_{k+1}\} \mid Y \in SM(T^k(M, U)), \text{ and } Y \cap U = X \cap U\}$;
- and
- the weight degrees of stable models have following relationships

$$W(T^{k+1}(M, U), Y) = \begin{cases} W(T^k(M, U), Y) * e^{2\alpha} & \text{if } Y \cap U \neq X \cap U, \\ W(T^k(M, U), Y) * e^\alpha & \text{otherwise.} \end{cases} \quad (8)$$

and for two stable models Y and Z of $T^k(M, U)$, if $Y \cap U = Z \cap U$, then $W(T^k(M, U), Y) = W(T^k(M, U), Z)$.

Example 2. Let L be the LP^{MLN} program in Example 1, and a set of literals $U = \{a, b\}$. By Definition 9, $T^0(L, U) = L \cup \{\alpha : a. \alpha : b.\}$, it is easy to check that all subsets of U are the stable models of $T^0(L, U)$, U is the unique probabilistic stable model. By Definition 8, $R(U, \emptyset, c_1)$ is as follows

$$\alpha : \leftarrow a, b, c_1. \quad (9)$$

$$\alpha : c_1 \leftarrow a, b. \quad (10)$$

and we have $T^1(L, U) = T^0(L, U) \cup R(U, \emptyset, c_1)$. The stable models and their weight degrees of L , $T^0(L, U)$, and $T^1(L, U)$ are shown in Table 1. From the table, we can observe that the flattening extension can be used to adjust the sets of literals that satisfy the most hard rules.

Lemma 4. Let L and M be two p -strongly equivalent LP^{MLN} programs, and $U = \text{lit}(\overline{L \cup M})$. For two consistent subsets X and Y of U , there exists a flattening extension $T^k(L, U)$ such that X and Y are probabilistic stable models of $T^k(L, U)$.

Lemma 4 provides a method to construct a set of necessary extensions of two p -strongly equivalent LP^{MLN} programs by constructing a set of flattening extensions, which means the PSE-condition is necessary to characterize the p -strong equivalence for LP^{MLN} programs.

Table 1: Computing Results in Example 2

Weight	\emptyset	$\{a\}$	$\{b\}$	$\{a, b\}$	$\{a, b, c_1\}$
L	e^0	$e^{\alpha+1}$	$e^{\alpha+1}$	-	-
$T^0(L, U)$	e^0	$e^{2\alpha+1}$	$e^{2\alpha+1}$	$e^{3\alpha+1}$	-
$T^1(L, U)$	$e^{2\alpha}$	$e^{4\alpha+1}$	$e^{4\alpha+1}$	$e^{4\alpha+1}$	$e^{4\alpha+1}$

Theorem 1. Let L and M be two LP^{MLN} programs,

- (i) L and M are p -strongly equivalent iff $LSE(L) = LSE(M)$, and there exist two constants c and k such that for each SE-model $(X, Y) \in LSE(L)$, $W(L, (X, Y)) = \exp(c + k * \alpha) * W(M, (X, Y))$;
- (ii) L and M are w -strongly equivalent iff they are p -strongly equivalent and the constants $c = k = 0$.

Example 3. Consider LP^{MLN} programs $L = \{w_1 : a \vee b. w_2 : b \leftarrow a.\}$ and $M = \{w_3 : b. w_4 : a \leftarrow \text{not } b.\}$, where w_i ($1 \leq i \leq 4$) is a variable denoting the weight of corresponding rule. It is easy to check that $(\{b\}, \{a, b\})$ is the unique non-total SE-model of L and M , therefore, L and M are semi-strongly equivalent. If the programs are also p -strongly equivalent, we have following system of linear equations, where $\mathcal{C} = \exp(k * \alpha + c)$ and $U = \{a, b\}$.

$$\begin{cases} W(L, (\emptyset, \emptyset)) = W(M, (\emptyset, \emptyset)) * \mathcal{C} \\ W(L, (\{a\}, \{a\})) = W(M, (\{a\}, \{a\})) * \mathcal{C} \\ W(L, (\{b\}, \{b\})) = W(M, (\{b\}, \{b\})) * \mathcal{C} \\ W(L, (U, U)) = W(M, (U, U)) * \mathcal{C} \end{cases} \Rightarrow \begin{cases} w_2 = c + k * \alpha \\ w_1 = w_4 + c + k * \alpha \\ w_1 + w_2 = w_3 + w_4 + c + k * \alpha \end{cases} \quad (11)$$

Solve the system of equations, we have L and M are p -strongly equivalent iff $w_2 = w_3 = c + k * \alpha$ and $w_1 = w_4 + c + k * \alpha$; and they are w -strongly equivalent iff $w_2 = w_3 = 0$ and $w_1 = w_4$.

4 Simplifying LP^{MLN} Programs

The notions of strong equivalences can be used to study the simplifications of logic programs. Specifically, if LP^{MLN} program L and M are strongly equivalent, and the program M is easier to solve or more friendly for human, then L can be replaced by M . In this section, we investigate the simplifications of LP^{MLN} programs via using the notions of strong equivalences. In particular, we present an algorithm to simplify and solve LP^{MLN} programs based on strong equivalences firstly. Then, we present some syntactic conditions that guarantee the strong equivalence between a single LP^{MLN} rule and the empty set \emptyset , which can be used to check the strong equivalences efficiently.

Definition 10. An LP^{MLN} rule $w : r$ is called semi-valid, if $w : r$ is semi-strongly equivalent to \emptyset ; the rule is called valid, if $w : r$ is p -strong equivalent to \emptyset .

In Definition 10, we specify two kinds of LP^{MLN} rules w.r.t semi-strong and p -strong equivalences. Obviously, a valid LP^{MLN} rule can be eliminated from any LP^{MLN} programs, while a semi-valid LP^{MLN} rule cannot. By the definition, eliminating a semi-valid LP^{MLN} rule does not change the stable models of original programs, but changes the probability distributions of the stable models, which means it may change the probabilistic stable models of original programs.

Example 4. Consider three LP^{MLN} programs $L = \{\alpha : a \leftarrow a.\}$, $M = \{\alpha : \leftarrow a.\}$, and $N = \{1 : a.\}$. It is easy to check that rules in L and M are valid and semi-valid, respectively. Table 2 shows the stable models and their probability degrees of LP^{MLN} programs N , $L \cup N$, and $M \cup N$. It can be observed that

Table 2: Computing Results in Example 4

Stable Model X	$P(N, X)$	$P(L \cup N, X)$	$P(M \cup N, X)$
\emptyset	0.27	0.27	1
$\{a\}$	0.73	0.73	0

eliminating the rule of M from $M \cup N$ makes all stable models of $M \cup N$ probabilistic, which means semi-valid rules cannot be eliminated directly.

Algorithm 1 provides a framework to simplify and solve LP^{MLN} programs based on the notions of semi-valid and valid LP^{MLN} rules. Firstly, simplify an LP^{MLN} program M by removing all semi-valid and valid rules (line 2 - 8). Then, compute the stable models of the simplified LP^{MLN} program via using some existing LP^{MLN} solvers, such as LPMLN2ASP, LPMLN2MLN [8], and LPMLN-Models [19] ect. Finally, compute the probability degrees of the stable models w.r.t. the simplified program and all semi-valid rules (line 9 - 12). The correctness of the algorithm can be proved by corresponding definitions.

Algorithm 1: Simplify and Solve LP^{MLN} Programs

Input: an LP^{MLN} program M

Output: stable models of M and their probability degrees

```

1  $S = \emptyset, M' = M$  ;
2 foreach  $w : r \in M$  do
3   if  $w : r$  is valid then
4      $M' = M' - \{w : r\}$  ;
5   else
6     if  $w : r$  is semi-valid then
7        $S = S \cup \{w : r\}$  ;
8        $M' = M' - \{w : r\}$  ;
9  $SM(M) = SM(M') = call-lpmln-solver(M')$  ;
10 foreach  $X \in SM(M)$  do
11    $W'(M, X) = exp(\sum_{w:r \in M' \cup S \text{ and } X \models w:r} w)$  ;
12 Compute probability degrees for each stable model  $X$  by Equation (2) and  $W'(M, X)$  ;
13 return  $SM(M)$  and corresponding probability degrees

```

In Algorithm 1, a crucial problem is to decide whether an LP^{MLN} rule is valid or semi-valid. Theoretically, it can be done by checking the SE-models of a rule, however, the approach is highly complex

Table 3: Syntactic Conditions

Name	Definition	Strong Equivalence
TAUT	$h(r) \cap b^+(r) \neq \emptyset$	p, semi
CONTRA	$b^+(r) \cap b^-(r) \neq \emptyset$	p, semi
CONSTR1	$h(r) = \emptyset$	semi
CONSTR2	$h(r) \subseteq b^-(r)$	semi
CONSTR3	$h(r) = \emptyset, b^+(r) = \emptyset, \text{ and } b^-(r) = \emptyset$	p, semi

in computation. Therefore, we investigate the syntactic conditions for the problem. Table 3 shows five syntactic conditions for a rule r , where TAUT and CONTRA have been introduced to investigate the program simplification of ASP [14, 4], CONSTR1 means the rule r is a constraint, and CONSTR3 is a special case of CONSTR1. Rules satisfying CONSTR2 is usually used to eliminate constraints in ASP, for example, rule “ $\leftarrow a.$ ” is equivalent to rule “ $p \leftarrow a, \text{not } p.$ ”, if the atom p does not occur in other rules. Based on these conditions, we present the characterization of semi-valid and valid LP^{MLN} rules.

Theorem 2. *An LP^{MLN} rule $w : r$ is semi-valid, iff the rule satisfies one of TAUT, CONTRA, CONSTR1 and CONSTR2.*

Theorem 3. *An LP^{MLN} rule $w : r$ is valid, iff one of following condition is satisfied*

- rule $w : r$ satisfies one of TAUT, CONTRA, and CONSTR3; or
- rule $w : r$ satisfies CONSTR1 or CONSTR2, and $w = 0$.

Theorem 2 and Theorem 3 can be proven by Lemma 1 and Theorem 1. It is worthy noting that conditions CONSTR1 and CONSTR2 means the only effect of constraints in LP^{MLN} is to change the probability distribution of inference results, which can also be observed in Example 2. In this sense, the constraints in LP^{MLN} can be regarded as the weak constraints in ASP, and Algorithm 1 is similar to the algorithm of solving ASP containing weak constraints. In both of algorithms, stable models are computed by removing (weak) constraints, and the certainty evaluations of the stable models are computed by combining these constraints.

Combining Theorem 2 and Theorem 3, Algorithm 1 is an alternative approach to enhance LP^{MLN} solvers. In addition, Theorem 2 and Theorem 3 also contribute to the field of knowledge acquiring. On the one hand, although it is impossible that rules of the form TAUT, CONTRA, and CONSTR3 are constructed by a skillful knowledge engineer, these rules may be obtained from data via rule learning. Therefore, we can use TAUT, CONTRA, and CONSTR3 as the heuristic information to improve the results of rule learning. On the other hand, CONSTR1 and CONSTR2 imply a kind of methodology of problem modeling in LP^{MLN} , that is, we can encode objects and relations by LP^{MLN} rules and facts, and adjust the certainty degrees of inference results by LP^{MLN} constraints. In fact, this is the core idea of ASP with weak constraints, LP^{MLN} is more flexible by contrast, since LP^{MLN} provides weak facts and rules besides weak constraints.

5 Conclusion and Future Work

In this paper, we present four kinds of notions of strong equivalences between LP^{MLN} programs by comparing the certainty degrees of stable models in different ways, i.e. semi-strong, w-strong and p-strong equivalences, where w-strong equivalence is the strongest notion, and semi-strong equivalence is the weakest notion. For each notion, we present a sufficient and necessary condition to characterize it, which can be viewed as a generalization of SE-model approach in ASP. After that, we present a sufficient and necessary condition that guarantees the strong equivalence between a single LP^{MLN} rule and the empty set, and we present an algorithm to simplify and solve LP^{MLN} programs by using the condition. The condition can also be used to improve the knowledge acquiring and increase the understanding of the methodology of problems modeling in LP^{MLN} .

As we showed in the paper, there is a close relationship between LP^{MLN} and ASP, especially, the constraints in LP^{MLN} can be regarded as the weak constraints in ASP. Concerning related work, the strong equivalence for ASP programs with weak constraints (abbreviated to ASP^{wc}) has been investigated [3].

It is easy to observe that the strong equivalence and corresponding characterizations of ASP^{wc} can be viewed as a special case of the p-strong equivalence in ASP.

For the future, we plan to improve the equivalences checking in the paper, and use these technologies to enhance LP^{MLN} solvers. And we also plan to extend the strong equivalence discovering method introduced in [13] to LP^{MLN} , which would help us to decide strong equivalence via some syntactic conditions.

6 Acknowledgments

We are grateful to the anonymous referees for their useful comments. The work was supported by the National Key Research and Development Plan of China (Grant No.2017YFB1002801).

References

- [1] Evgenii Balai & Michael Gelfond (2016): *On the Relationship between P-log and LP^{MLN}* . In Subbarao Kambhampati, editor: *Proceedings of the 25th International Joint Conference on Artificial Intelligence*, pp. 915–921.
- [2] Gerhard Brewka, Thomas Eiter & Mirosław Truszczyński (2011): *Answer Set Programming at a Glance*. *Communications of the ACM* 54(12), pp. 92–103, doi:10.1145/2043174.2043195.
- [3] Thomas Eiter, Wolfgang Faber, Michael Fink & Stefan Woltran (2007): *Complexity results for answer set programming with bounded predicate arities and implications*. *Annals of Mathematics and Artificial Intelligence* 51(2-4), pp. 123–165, doi:10.1007/s10472-008-9086-5.
- [4] Thomas Eiter, Michael Fink, Hans Tompits & Stefan Woltran (2004): *Simplifying Logic Programs Under Uniform and Strong Equivalence*. In: *Proceedings of the 7th International Conference on Logic Programming and Nonmonotonic Reasoning*, pp. 87–99, doi:10.1007/978-3-540-24609-1_10.
- [5] Thomas Eiter & Tobias Kaminski (2016): *Exploiting Contextual Knowledge for Hybrid Classification of Visual Objects*. In Jürgen Dix, Luís Fariñas del Cerro & Ulrich Furbach, editors: *Proceedings of the 15th European Conference on Logics in Artificial Intelligence, Lecture Notes in Computer Science 10021*, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 223–239, doi:10.1007/978-3-319-48758-8_15.
- [6] Michael Gelfond & Vladimir Lifschitz (1988): *The Stable Model Semantics for Logic Programming*. In Robert A. Kowalski & Kenneth A. Bowen, editors: *Proceedings of the Fifth International Conference and Symposium on Logic Programming*, MIT Press, pp. 1070–1080.
- [7] Katsumi Inoue & Chiaki Sakama (2004): *Equivalence of Logic Programs Under Updates*. In: *Proceedings of the 9th European Workshop on Logics in Artificial Intelligence*, 3229, pp. 174–186, doi:10.1007/978-3-540-30227-8_17.
- [8] Joohyung Lee, Samidh Talsania & Yi Wang (2017): *Computing LP^{MLN} using ASP and MLN solvers*. *Theory and Practice of Logic Programming* 17(5-6), pp. 942–960, doi:10.1017/S1471068417000400.
- [9] Joohyung Lee & Yi Wang (2016): *Weighted Rules under the Stable Model Semantics*. In Chitta Baral, James P. Delgrande & Frank Wolter, editors: *Proceedings of the Fifteenth International Conference on Principles of Knowledge Representation and Reasoning*, AAAI Press, pp. 145–154.
- [10] Joohyung Lee & Yi Wang (2018): *Weight Learning in a Probabilistic Extension of Answer Set Programs*. In: *Proceedings of the 16th International Conference on the Principles of Knowledge Representation and Reasoning*, pp. 22–31.
- [11] Joohyung Lee & Zhun Yang (2017): *LP^{MLN} , Weak Constraints, and P-log*. In Satinder P. Singh & Shaul Markovitch, editors: *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence*, AAAI Press, pp. 1170–1177.

- [12] Valdimir Lifschitz, David Pearce & Agustín Valverde (2001): *Strongly equivalent logic programs*. *ACM Transactions on Computational Logic* 2(4), pp. 526–541, doi:10.1145/383779.383783.
- [13] Fangzhen Lin & Yin Chen (2007): *Discovering Classes of Strongly Equivalent Logic Programs*. *Journal of Artificial Intelligence Research* 28, pp. 431–451, doi:10.1613/jair.2131.
- [14] Mauricio Osorio, Juan Antonio Navarro & José Arrazola (2001): *Equivalence in Answer Set Programming*. In: *Proceedings of the 11th International Workshop on Logic Based Program Synthesis and Transformation*, pp. 57–75, doi:10.1007/3-540-45607-4_4.
- [15] Matthew Richardson & Pedro Domingos (2006): *Markov logic networks*. *Machine Learning* 62(1-2), pp. 107–136, doi:10.1007/s10994-006-5833-1.
- [16] Hudson Turner (2001): *Strong Equivalence for Logic Programs and Default Theories (Made Easy)*. In: *Proceedings of the 6th International Conference on Logic Programming and Nonmonotonic Reasoning*, pp. 81–92, doi:10.1007/3-540-45402-0_6.
- [17] Bin Wang & Zhizheng Zhang (2017): *A Parallel LP^{MLN} Solver: Primary Report*. In Bart Bogaerts & Amelia Harrison, editors: *Proceedings of the 10th Workshop on Answer Set Programming and Other Computing Paradigms*, CEUR-WS, Espoo, Finland, pp. 1–14.
- [18] Bin Wang, Zhizheng Zhang, Hongxiang Xu & Jun Shen (2018): *Splitting an LP^{MLN} Program*. In: *Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence*, pp. 1997–2004.
- [19] Wei Wu, Hongxiang Xu, Shutao Zhang, Jiaqi Duan, Bin Wang, Zhizheng Zhang, Chenglong He & Shiqiang Zong (2018): *LPMLNModels: A Parallel Solver for LPMLN*. In: *2018 IEEE 30th International Conference on Tools with Artificial Intelligence (ICTAI)*, IEEE, pp. 794–799, doi:10.1109/ICTAI.2018.00124.