Proof Nets, Coends and the Yoneda Isomorphism

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Proof nets provide permutation-independent representations of proofs and are used to investigate coherence problems for monoidal categories. We investigate a coherence problem concerning Second Order Multiplicative Linear Logic (MLL2), that is, the one of characterizing the equivalence over proofs generated by the interpretation of quantifiers by means of ends and coends.

We provide a compact representation of proof nets for a fragment of MLL2 related to the Yoneda isomorphism. By adapting the "rewiring approach" used in coherence results for *-autonomous categories, we define an equivalence relation over proof nets called "rewitnessing". We prove that this relation characterizes, in this fragment, the equivalence generated by coends.

1 Introduction

Proof nets are usually investigated as canonical representations of proofs. For the proof-theorist, the adjective "canonical" indicates a representation of proofs insensitive to admissible permutations of rules; for the category-theorist, it indicates a faithful representation of arrows in free monoidal categories (e.g. *-autonomous categories), by which coherence results can be obtained.

This twofold approach has been developed extensively in the case of Multiplicative Linear Logic (see for instance [5, 6]). The use of MLL proof nets to investigate coherence problems relies on the correspondence between proof nets and a particular class of dinatural transformations (see [5]). As dinatural transformations provide a well-known interpretation of parametric polymorphism (see [1, 16]), it is natural to consider the extension of this correspondence to second order Multiplicative Linear Logic MLL2. This means investigating the "coherence problem" generated by the interpretation of quantifiers as ends/coends, that is, to look for a faithful proof net representation of coends over a *-autonomous category.

The main difficulty of this extension is that, as is well-known, dinaturality does not scale to second order (e.g. System F, see [26]): the dinatural interpretation of proofs generates an equivalence over proofs which strictly extends the equivalence generated by β and η conversions. In particular, coends induce "generalized permutations" of rules ([36]) to which neither System F proofs nor standard proof nets for MLL2 are insensitive. For instance, the interpretation of quantifiers as ends/coends (whose definition is recalled in appendix A) equates the distinct System F derivations in fig. 1a as well as the distinct proof nets in fig. 1b. From these examples it can be seen that such generalized permutations do not preserve the witnesses of existential quantification (or, equivalently, of the elimination of universal quantification).

Several well-known issues in the System F representation of categorial structures can be related to this phenomenon. For instance, the failure of universality for the "Russell-Prawitz" translation of connectives (e.g. the failure of the isomorphism $A\otimes B\simeq \forall X((A\multimap B\multimap X)\multimap X)$), and the failure of initiality for the System F representation of initial algebras (i.e. the failure of the isomorphism $\mu X.T(X)\simeq$

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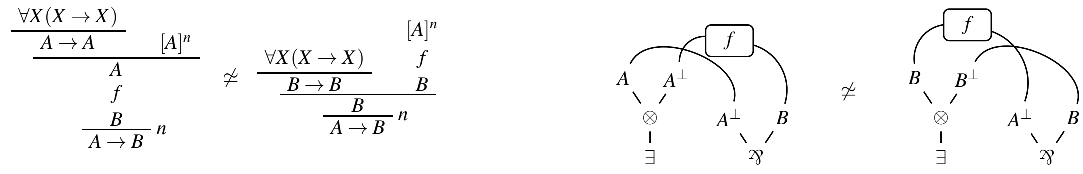


(a) Failure of dinaturality in System F

(b) Failure of dinaturality for proof nets

Figure 1: Failure of dinaturality in System F and MLL2

$\forall X((T(X) \Rightarrow X) \Rightarrow X))$. In such cases, the failure is solved by considering proofs modulo the equivalence induced by dinaturality (see [33, 17]). All these can be seen as instances of a more general problem, namely the fact that the *Yoneda isomorphism* $Nat(\mathbb{C}(a,x),F) \simeq F(a)$ corresponds, in the language of MLL2, to a series of logical equivalences of the form $\forall X((A \multimap X) \multimap F[X])) \simeq F[A/X]$ which fail to be isomorphisms of types. In this paper we investigate the possibility to provide a faithful representation of the Yoneda isomorphism, and more generally of ends and coends, by means of MLL2 proof nets.

As a consequence of the isomorphism $\forall X(X \multimap X) \simeq \mathbf{1}$, which is a particular instance of the Yoneda isomorphism just recalled, the proof net representation of quantifiers as ends and coends must include a faithful representation of multiplicative units. From this we can deduce some *a priori* limitations to our enterprise: it is well-known that no canonical representation of MLL with multiplicative units can have both a tractable correctness criterion and a tractable translation from sequent calculus ([18]). However, in usual approaches to multiplicative units proof nets are considered modulo an equivalence relation called *rewiring* ([37, 6, 22]), which provides a partial solution to this problem. The "rewiring approach" ([22]) allows to circumvent the complexity of checking arrows equivalence in the free *-autonomous category by isolating the complex part into a geometrically intuitive equivalence relation.

We define a compact representation of proof nets (called $\exists$-linkings) for the fragment of MLL2 which adapts the rewiring technique to second order quantification. We consider the system $\mathsf{MLL2}_{\mathscr{Y}}$, in which quantification $\forall XA$ is restricted to "Yoneda formulas", i.e. formulas of the form $\forall X((\bigotimes_i^n C_i \multimap X) \multimap D[X])$. This fragment contains the multiplicative "Russell-Prawitz" formulas as well as the translation of multiplicative units. In our approach rewiring is replaced by *rewitnessing*, an equivalence relation which allows to rename the witnesses of existential quantifiers. This approach is related to rewiring in the sense that, when restricted to the second order translation of units, $\exists$-linkings correspond exactly to the "lax linkings" in [22].

Our main result (theorem 2) is that the equivalence over proofs generated by coends coincides exactly with the rewitnessing equivalence over $\exists$-linkings. More precisely, we define an equivalence $\simeq_\varepsilon$ over standard MLL2 proof nets, where two proof nets are equivalent when their interpretations in any dinatural model coincide, and we show that, within the fragment $\mathsf{MLL2}_{\mathscr{Y}}$, $\pi \simeq_\varepsilon \pi'$ holds iff the associated $\exists$-linkings ℓ_π and $\ell_{\pi'}$ are equivalent up to rewitnessing. To prove this, we construct an isomorphism between the category generated by MLL2 proof nets modulo the equivalence induced by dinaturality and the category generated by $\exists$-linking modulo rewitnessing. The proof that this is an isomorphism will essentially rely on the "true" Yoneda isomorphism. These results imply that $\exists$-linkings form a *-autonomous category in which $\forall X(X \multimap X)$ is the tensor unit and provide a faithful representation of coends.

In the category of $\exists$-linkings the Yoneda isomorphism is a true isomorphism and the "Russell-Prawitz" isomorphisms like $A \otimes B \simeq \forall X((A \multimap B \multimap X) \multimap X)$ hold. The representation of initial algebras falls outside the scope of the fragment $\mathsf{MLL}_{\mathscr{Y}}$, due to the more complex shape of the formulas involved.

However, following the ideas in [38], a generalization of the approach here presented might yield similar results for the representation of initial algebras.

Related work Dinaturality is a well-investigated property of System F and is usually related to parametric polymorphism (see [1, 33]). The connections between dinaturality, coherence and proof nets are well-investigated in the case of MLL, with or without units ([4, 5, 6, 24, 22, 19, 30, 20]). An extensive literature exists on coends in monoidal categories (see [27] for a survey). String diagram representations of some coends can be found in the literature on Hopf algebras and their application to conformal field theory ([23, 12]). Such coends are all of the restricted form considered in this paper and their representation seems comparable to the one here proposed. A different approach to quantifiers as ends/coends over a symmetric monoidal closed category appears in [31], through a bifibrational reformulation of the Lawvere's presheaf hyperdoctrine in the 2-category of distributors.

The universality problem for the "Russell-Prawitz" translation is related to the *instantiation overflow* property ([10]), by which one can transform the System F proofs obtained by this translation into proofs in F_{at} or *atomic* System F, which have the desired properties (see [9]). In [32] is shown that the atomized proofs are equivalent to the original ones modulo dinaturality. $\exists$-linkings provide a very simple approach to instantiation overflow, to be investigated in the future, as the transformation from F to F_{at} corresponds to rewitnessing.

The representation of proof nets here adopted is inspired from results on MLL with units ([37, 6, 22]) and on $\mathsf{MLL1}$ ([21]). Proof nets for first-order and second order quantifers were first conceived by means of boxes ([13]). Later, Girard proposed two distinct boxes-free formalisms (in [14, 15] for $\mathsf{MLL1}$ but extendable to $\mathsf{MLL2}$, see [8]), the second of which is referred here as "Girard nets". Different refinements of proof nets for $\mathsf{MLL1}$ and $\mathsf{MLL2}$ have been proposed ([29, 21] for $\mathsf{MLL1}$ and [35] for $\mathsf{MLL2}$) to investigate variable dependency issues related to Herbrand theorem and unification, which are not considered here.

2 Girard nets and their interpretation in dinatural models

We let $\mathscr{L}^2$ be the language generated by a countable set of variables $X,Y,Z,\dots \in \mathtt{Var}$ and their negations $X^{\perp},Y^{\perp},Z^{\perp},\dots$ and the connectives $\otimes,⅋,\forall,\exists$. Negation is extended in an obvious way into an equivalence relation over formulas. By sequents $\Gamma,\Delta,\dots$ we indicate finite multisets of formulas. A sequent Γ is *clean* when no variable occurs both free and bound in Γ and any variable in Γ is bound by at most one $\forall$ or $\exists$ connective.

By $\mathsf{MLL2}$ we indicate the standard sequent calculus over $\mathscr{L}^2$. [15] describes proof nets for first-order MLL. Both the description of proof structures and the correctness criterion can be straightforwardly turned into a definition of proof structures and proof nets for $\mathsf{MLL2}$ (see for instance [8]). We indicate the latter as *Girard proof structures* and *Girard nets* (shortly, *G*-proof structures and *G*-nets[1]). We let $\mathbb{G}$ indicate the *category of G-nets*, whose objects are the types of $\mathsf{MLL2}$ and where $\mathbb{G}(A,B)$ is the set of cut-free *G*-nets of conclusions $A^{\perp},B$ (with composition given by cut-elimination).

[1]In [15] the definition of proof structures is based on two conditions: (1) that any $\forall$ link has a distinct eigenvariable and (2) that the conclusions of a proof structures have no free variable (in particular, new constants $\bar{x}$ are introduced to eliminate free variables). Moreover, in the definition of the correctness criterion any $\forall$-link of eigenvariable X can *jump* on any formula in which X occurs free. In [21] conditions (1) and (2) are replaced by the equivalent condition that the conclusions of the proof structure plus the witnesses of existential links must form a *clean* sequent and the correctness criterion is modified by demanding that a $\forall$-link of eigenvariable X can *jump* on any $\exists$-link whose witness formula contains free occurrences of X. Here we will consider this formulation.

Some useful definitions and properties of *-autonomous categories and coends can be found in appendix A. It is well-known (see [25]) that, if we let $\mathbb{P}$ be the category of MLL proof nets and $\mathbb{C}$ be any (strict) *-autonomous category, then any map $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$ generates a functor $\Phi : \mathbb{P} \to \mathbb{C}$. We will now extend this result to MLL2 by considering *dinatural models*, that is, models in which MLL2 proofs are interpreted as dinatural transformations [1]. We show how any G-net can be interpreted in a dinatural model over a *-autonomous category $\mathbb{C}$, and we deduce that any map $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$ generates a functor $\Phi : \mathbb{G} \to \mathbb{C}$.

It is well-known that dinatural transformations do not compose. The standard approach to interpret second order proofs (see [1]) is thus to restrict to a class of composable dinatural transformations. In order to interpret quantifiers one considers then *relativized* ends/coends, i.e. wedges/co-wedges (see appendix A) which are universal among the class of dinatural transformations in the model.

Definition 1 (dinatural model)**.** *Let $\mathbb{C}$ to be a (strict) *-autonomous category $\mathbb{C}$. A* dinatural model over *$\mathbb{C}$ is a category $\mathscr{F}$ such that*

- *the objects of $\mathscr{F}$ are multi-variant functors over $\mathbb{C}$, including projections of any arity and the constant functor $\mathbf{1}^{\mathbb{C}}$, and closed with respect to $\otimes$ and *;*
- *for all objects F,G, $\mathscr{F}(F,G)$ is a set of dinatural transformations from F to G, so that $\mathscr{F}$ is *-autonomous with unit $\mathbf{1}^{\mathbb{C}}$, monoidal product $\otimes$ and involution *;*
- *the objects of $\mathscr{F}$ contain all ends and coends relativized to arrows in $\mathscr{F}$.*

The definition above can be recast in the standard fibrational setting of second order models (see [34]) by using properties of ends and coends. Two dinatural models are suggested in [5] and [3]. Moreover, a *free* dinatural model is obtained by quotienting the syntactic model of MLL2 under the congruence generated by all equations expressing the fact that quantifiers correspond to wedges and co-wedges.

In the rest of this section we suppose given a dinatural model $\mathscr{F}$ over a (strict) *-autonomous category $\mathbb{C}$. Any formula $A \in \mathscr{L}^2$ whose free variables are within $X_1,\dots,X_n$ can be interpreted as a functor $A^{\mathbb{C},\mathscr{F}} : (\mathbb{C}^{op} \times \mathbb{C})^n \to \mathbb{C}$ in $\mathscr{F}$ by letting

$$X_i^{\mathbb{C},\mathscr{F}}(\vec{a},\vec{b}) := b_i \qquad X_i^{\mathbb{C},\mathscr{F}}(\vec{f},\vec{g}) := g_i$$
$$(A\otimes B)^{\mathbb{C},\mathscr{F}} := A^{\mathbb{C},\mathscr{F}} \otimes B^{\mathbb{C},\mathscr{F}} \quad (\forall YA)^{\mathbb{C},\mathscr{F}} := \int_y^{\mathscr{F}} A^{\mathbb{C},\mathscr{F}}(y,y) \quad (A^{\perp})^{\mathbb{C},\mathscr{F}} := (A^{\mathbb{C},\mathscr{F}})^*$$

where $\int_y^{\mathscr{F}} F$ indicates the end relativized to $\mathscr{F}$. In the following lines, since reference to $\mathscr{F}$ is clear, we will write $A^{\mathbb{C},\mathscr{F}}$ as $A^{\mathbb{C}}$ and $\int_y^{\mathscr{F}} F$ as $\int_y F$ for simplicity. For a clean sequent $\Gamma = A_1,\dots,A_n$, whose free variables are within $X_1,\dots,X_n$, we let $\Gamma^{\mathbb{C}} := A_1^{\mathbb{C}} \parr \cdots \parr A_n^{\mathbb{C}}$ (where $x \parr y := \mathbb{C}(x^{\perp},y)$) if $n \geq 1$ and $\Gamma^{\mathbb{C}} = \mathbf{1}_{\mathbb{C}}$ if $n = 0$.

Lemma 1 (substitution lemma)**.** $(A[B/X])^{\mathbb{C}}(x,x) = A^{\mathbb{C}}(B^{\mathbb{C}}(x,x), B^{\mathbb{C}}(x,x))$.

Proof. Induction on A. The only delicate case is $A = \forall YA'$, and, as we can suppose that $B^{\mathbb{C}}$ does not depend on y, $(A[B/X])^{\mathbb{C}}(x,x) = \int_y ((A'[B/X])^{\mathbb{C}}((y,x),(y,x))) \stackrel{[i.h.]}{=} \int_y (A')^{\mathbb{C}}((y,B^{\mathbb{C}}),(y,B^{\mathbb{C}})) = (\int_y (A')^{\mathbb{C}}((y,x),(y,x)))(B^{\mathbb{C}},B^{\mathbb{C}}) = A^{\mathbb{C}}(B^{\mathbb{C}},B^{\mathbb{C}})$.

$\square$

Let π be a cut-free G-net of conclusions Γ and let all formulas occurring in π be within $X_1,\dots,X_n$. We now show that π can be interpreted as a dinatural transformation $\pi^{\mathbb{C},\mathscr{F}} : \mathbf{1}_{\mathbb{C}} \to \Gamma^{\mathbb{C},\mathscr{F}}$[2]. As in the case

[2] As explained in appendix A, we omit for readability reference to variables $x_1,\dots,x_n$.

of functors, since reference to $\mathscr{F}$ is clear, we will simply write $\pi^{\mathbb{C},\mathscr{F}}$ as $\pi^{\mathbb{C}}$. Similarly to [25] (Th. 2.3.1. p. 32), we can define $\pi^{\mathbb{C}}$ by induction on a sequentialization of π. We adopt a sequentialization theorem for G-nets inspired from [21] and described in appendix B.

- if π is an axiom link of conclusions $X^{\perp}, X$, then $\pi^{\mathbb{C}} := \hat{\mathbf{1}}_{A^{\mathbb{C}}}$.
- if $\Gamma = \Delta, A \parr B$ and π is obtained from a π' of conclusions Δ, A, B by adding a $\parr$-link, then $\pi^{\mathbb{C}} := (\pi')^{\mathbb{C}}$.
- if $\Gamma = \Delta_1, \Delta_2, A \otimes B$ and π is obtained from π_1 of conclusions Δ_1, A and π_2 of conclusions Δ_2, B, then $\pi^{\mathbb{C}} := t_{\vec{x}} \circ \big((\pi_1)^{\mathbb{C}} \otimes (\pi_2)^{\mathbb{C}}\big)$, where $t_{\vec{x}} : (\Delta_1^{\mathbb{C}} \parr A^{\mathbb{C}}) \otimes (\Delta_2^{\mathbb{C}} \parr B^{\mathbb{C}}) \to \Delta_1^{\mathbb{C}} \parr \Delta_2^{\mathbb{C}} \parr (A \otimes B)^{\mathbb{C}}$ is $\iota_{A^{\mathbb{C}}, \Delta_1^{\mathbb{C}}, (\Delta_2 \parr B)^{\mathbb{C}}} \circ (\iota_{A^{\mathbb{C}}, \Delta_2^{\mathbb{C}}, B^{\mathbb{C}}} \parr B^{\mathbb{C}})$, given the natural transformation $\iota_{a,b,c} : (a \parr b) \otimes c \to (a \otimes c) \parr b$.
- if $\Gamma = \Delta, \forall Y A$ and π is obtained from π' of conclusions Δ, A, then from $(\pi')^{\mathbb{C}}_x : \mathbf{1}_{\mathbb{C}} \to \Delta^{\mathbb{C}} \parr A^{\mathbb{C}}$ we obtain (by applying the natural isomorphism $\mathbb{C}(a \otimes b^{\perp}, c) \simeq \mathbb{C}(a, b \parr c)$) a dinatural transformation $\theta_x : (\Delta^{\mathbb{C}})^{\perp} \to A^{\mathbb{C}}$ [3]. $\pi^{\mathbb{C}}$ is now obtained by the universality of (relativized) ends, as shown by the diagram below:

$$\begin{tikzcd}
(\Delta^{\mathbb{C}})^{\perp} \arrow[dr, dashed, "\pi^{\mathbb{C}}"] \arrow[drr, bend left, "\theta_a"] \arrow[ddr, bend right, "\theta_b"'] & & \\
& \int_y A^{\mathbb{C}}(y,y) \arrow[r, "\delta_a^{A^{\mathbb{C}}}"] \arrow[d, "\delta_b^{A^{\mathbb{C}}}"] & A^{\mathbb{C}}(a,a) \arrow[d, "A^{\mathbb{C}}(a,f)"] \\
& A^{\mathbb{C}}(b,b) \arrow[r, "A^{\mathbb{C}}(f,b)"'] & A^{\mathbb{C}}(a,b)
\end{tikzcd}$$

- if $\Gamma = \Delta, \exists Y A$ and π is obtained from π' of conclusions $\Delta, A[B/X]$, then $\pi^{\mathbb{C}}$ is obtained from $(\pi')^{\mathbb{C}}$ by the chain of arrows below (by exploiting lemma 1):

$$\mathbf{1}_{\mathbb{C}} \xrightarrow{(\pi')^{\mathbb{C}}} \Delta^{\mathbb{C}} \parr A^{\mathbb{C}}(B^{\mathbb{C}}, B^{\mathbb{C}}) \xrightarrow{\omega_{B^{\mathbb{C}}}^{\Delta^{\mathbb{C}} \parr A^{\mathbb{C}}}} \int^x (\Delta^{\mathbb{C}} \parr A^{\mathbb{C}}(x,x)) \xrightarrow{\quad\nu\quad} \Delta^{\mathbb{C}} \parr \int^x A^{\mathbb{C}}(x,x)$$

where ν is given in equation A.5 in appendix A.

Remark 1. *It is well-known that* MLL *proof nets can be interpreted as (composable) dinatural transformations over any* **-autonomous category* $\mathbb{C}$ *[5], without requiring a dinatural model over* $\mathbb{C}$ *to exist. This fact does not seem to scale to* MLL2*, since the last step of the definition above exploits the composition of two dinatural transformations.*

We show now that the definition of $\pi^{\mathbb{C}}$ does not depend on the sequentialization chosen. We must consider all possible permutations of rules in a sequentialization of $\pi^{\mathbb{C}}$. We call a $\exists$ link *simple* if it has no incoming jump. For readability we will often confuse formulas A and proof nets π with their interpretations $A^{\mathbb{C}}$ and $\pi^{\mathbb{C}}$.

- permutations between $\parr, \forall$ and simple $\exists$:

  ($\parr/\parr$) We can argue as in [25].

  ($\parr/\forall$) π_1, π_2, of conclusions $\Gamma, A \parr B, \forall X C$ come from π' of conclusions Γ, A, B, C. The claim follows from the fact that the introduction of $\parr$ does not change the interpretation.

  ($\forall/\forall$) π_1, π_2, of conclusions $\Gamma, \forall X A, \forall Y B$ come from π' of conclusions Γ, A, B. The claim follows from $\int_x A^{\mathbb{C}}(x,x) \parr \int_y B^{\mathbb{C}}(y,y) \overset{Eq.\,A.1}{\simeq} \int_x \int_y (A^{\mathbb{C}}(x,x) \parr B^{\mathbb{C}}(y,y)) \overset{Eq.\,A.3}{\simeq} \int_y \int_x (A^{\mathbb{C}}(x,x) \parr B^{\mathbb{C}}(y,y)) \overset{Eq.\,A.1}{\simeq} \int_x A^{\mathbb{C}}(x,x) \parr \int_y B^{\mathbb{C}}(y,y)$.

  ($\parr/\exists$) Similar to case ($\parr/\forall$).

[3] More precisely, θ_x is $\theta_{x_1,\dots,x_n,x}$ and comes from $(\pi')^{\mathbb{C}}_{x_1,\dots,x_n,x}$, where $(\Delta^{\mathbb{C}})^{\perp}$ does not depend on x.

(∀/∃) π_1, π_2 of conclusions $\forall XA, \exists YB$ (we omit contexts Γ for simplicity) come from π' of conclusions $A, B[C/Y]$, where C has no free occurrence of X. We let $c = C^{\mathbb{C}}$, θ indicate the translation of the G-net of conclusions $\forall XA, B[C/Y]$ and σ_x indicate the translation of the G-net of conclusions $A, \exists YB$, so that $\pi_1 = (\int_x A \parr \omega_c^B) \circ \theta$ and π_2 is the universality arrow in the dinaturality diagram for σ_x. Then $\pi_1 = \pi_2$ follows from the universality of π_2, as shown by the diagram below:

$$\begin{tikzcd}
\mathbf{1}_{\mathbb{C}} \arrow[dr, dashed, "\theta"] \arrow[drr, dashed, "\pi_2"] \arrow[rrr, "\pi'_a"] \arrow[drrrr, bend left, "\sigma_a"] \arrow[ddr, "\pi'_b"'] \arrow[ddddrr, bend right, "\sigma_b"'] & & & & \\
& \int_x A \parr B(c,c) \arrow[rr, "\delta^A_a \parr B(c,c)"] \arrow[dr, "\int_x A \parr \omega^B_c"] \arrow[dd, "\delta^A_b \parr B(c,c)"'] & & A(a,a) \parr B(c,c) \arrow[dr, "A \parr \omega^B_c"] \arrow[dd, "A(a,f) \parr B(c,c)"] & \\
& & \int_x (A \parr \int^y B) \simeq \int_x A \parr \int^y B \arrow[rr, "\delta^A_a \parr \int^y B"] \arrow[dd, "\delta^A_b \parr \int^y B"] & & A(a,a) \parr \int^y B \arrow[dd, "A(a,f) \parr \int^y B"] \\
& A(b,b) \parr B(c,c) \arrow[rr, "A(f,b) \parr B(c,c)"] \arrow[dr, "A \parr \omega^B_c"] & & A(a,b) \parr B(c,c) \arrow[dr, "A \parr \omega^B_c"] & \\
& & A(b,b) \parr \int^y B \arrow[rr, "A(f,b) \parr \int^y B"'] & & A(a,b) \parr \int^y B
\end{tikzcd}$$

(∃/∃) Similar to case (∀/∀).

- permutations between a splitting $\otimes$ and $\parr, \forall$ or simple $\exists$:

($\otimes/\parr$) We can argue as in [25].

($\otimes/\forall$) π_1, π_2, of conclusions $A \otimes C, \forall XB$ (we omit contexts Γ, Δ for simplicity) are obtained from σ, of conclusions A, B and τ, of conclusion C, so that $\pi_1 = \iota_{A, \int_x B, C} \circ (\int_x \sigma \otimes \tau)$, where $\int_x \sigma$ is the interpretation of the G-net obtained from σ by adding a $\forall$-link and π_2 is the universality arrow in the universality diagram for $\iota_{A,B(x),C} \circ (\sigma_x \otimes \tau)$. Then $\pi_1 = \pi_2$ follows from the universality of π_2, as shown by the diagram below.

$$\begin{tikzcd}
\mathbf{1}_{\mathbb{C}} \arrow[dr, "\int_x \sigma \otimes \tau"] \arrow[ddr, dashed, "\pi_2"] \arrow[drrr, bend left, "\sigma_a \otimes \tau"] \arrow[dddr, "\sigma_b \otimes \tau"'] & & & & \\
& (A \parr \int_x B) \otimes C \arrow[rr, "(A \parr \delta^B_a) \otimes C"] \arrow[dr, "\iota_{A, \int_x B, C}"] \arrow[dd, "(A \parr \delta^B_b) \otimes C"] & & (A \parr B(a,a)) \otimes C \arrow[dr, "\iota_{A,B(a,a),C}"] & \\
& & (A \otimes C) \parr \int_x B \simeq \int_x ((A \otimes C) \parr B) \arrow[rr, "(A \otimes C) \parr \delta^B_a"'] \arrow[dd, "(A \otimes C) \parr \delta^B_b"] & & (A \otimes C) \parr B(a,a) \arrow[dd, "(A \otimes C) \parr B(a,f)"] \\
& (A \parr B(b,b)) \otimes C \arrow[dr, "\iota_{A,B(b,b),C}"'] & & & \\
& & (A \otimes C) \parr B(b,b) \arrow[rr, "(A \otimes C) \parr B(f,b)"'] & & (A \otimes C) \parr B(a,b)
\end{tikzcd}$$

($\otimes/\exists$) π_1, π_2, of conclusions $A \otimes D, \exists XB$ (again, we omit contexts Γ, Δ for simplicity) are obtained from σ, of conclusions $A, B[C/X]$ and τ, of conclusions D, so that $\pi_1 = \iota_{A, \int^x B, D} \circ (\int^x \sigma \otimes \tau)$, where $c = C^{\mathbb{C}}$, $\int^x \sigma = (A \parr \omega_c^B) \circ \sigma$ is the interpretation of the G-net obtained from σ by adding a $\exists$-link and $\pi_2 = ((A \otimes D) \parr \omega_c^B) \circ \iota_{A,B(c,c),D} \circ (\sigma \otimes \tau)$. Then $\pi_1 = \pi_2$ follows from the naturality of ι, as shown in the diagram below.

$$\begin{tikzcd}
& (A \parr B[C/X]) \otimes D \arrow[r, "\iota_{A,B[C/X],D}"] \arrow[dd, "(A \parr \omega^B_c) \otimes D"'] & (A \otimes D) \parr B[C/X] \arrow[dd, "(A \otimes D) \parr \omega^B_c"] \\
\mathbf{1}_{\mathbb{C}} \arrow[ur, "\sigma \otimes \tau"] \arrow[dr, "(\int^x \sigma) \otimes \tau"'] & & \\
& (A \parr \int^x B) \otimes D \arrow[r, "\iota_{A, \int^x B, D}"'] & (A \otimes D) \parr \int^x B
\end{tikzcd}$$

- permutations between splitting $\otimes$: we can argue as in [25].

The definition above can be extended to the case of a G-net with cuts: if π has conclusions Γ and cut-formulas $B_1, \ldots, B_n$, then we can transform π into a G-net π_{cut} of conclusions $\Gamma, [B_1 \otimes B_1^{\perp}, \ldots, B_n \otimes B_n^{\perp}]$. Then we can define $\pi^{\mathbb{C}}$ as $(id_{\Gamma^{\mathbb{C}}} \parr \hat{\bot}_{B_1^{\mathbb{C}}} \parr \cdots \parr \hat{\bot}_{B_n^{\mathbb{C}}}) \circ \pi_{cut}^{\mathbb{C}}$. The following proposition shows that if the G-net π reduces to the cut-free G-net π_0, then $\pi^{\mathbb{C}} = \pi_0^{\mathbb{C}}$. Hence it shows that the denotation $\pi^{\mathbb{C}}$ is invariant with respect to reduction.

Proposition 1. *Let π be a G-net with cuts of conclusions Γ and π_0 be the G-net obtained from π by eliminating all cuts. Then $\pi^{\mathbb{C}} = \pi_0^{\mathbb{C}}$.*

Proof. We consider a reduction sequence of π which follows a sequentialization, hence such that any time a cut is eliminated, this cut corresponds to a splitting tensor of π. As this reduction sequence is finite and terminates on π_0 (by strong normalization and confluence), we can argue by induction on its length. The cases of MLL cuts can be treated by arguing as in the proof of Lemma 2.3.4, p. 36, of [25]. We consider then the case of a cut $\forall/\exists$. Let π be a G-net of conclusions $\Gamma, [\forall XA \otimes \exists XA^{\perp}]$ and let π' be the G-net of conclusions $\Gamma, [A[B/X] \otimes A^{\perp}[B/X]]$ obtained by applying one reduction step to π'. We must show that $\sigma_1 = (\Gamma^{\mathbb{C}} \parr \hat{\bot}_{\int_x A^{\mathbb{C}}(x,x)}) \circ \pi^{\mathbb{C}}$ is equal to $\sigma_2 = (\Gamma^{\mathbb{C}} \parr \hat{\bot}_{A^{\mathbb{C}}(b,b)}) \circ (\pi')^{\mathbb{C}}$, where $b = B^{\mathbb{C}}$. Since the $\otimes$-link is splitting, $\Gamma = \Gamma_1, \Gamma_2$ and π (resp. π') splits into π_1 of conclusions $\Gamma_1, \forall XA$ (resp π_1' of conclusions $\Gamma_1, A[B/X]$) and π_2 of conclusions $\Gamma_2, \exists XA^{\perp}$ (resp. π_2' of conclusions $\Gamma_2, A^{\perp}[B/X]$). The claim follows then from the induction hypothesis and the commutation of the diagram below, which is a consequence of the dinaturality of $\hat{\bot}_x$ and of the fact that $\omega_b^{A^{\perp}} = (\delta_b^A)^{\perp}$ (as before, for readability we confuse formulas A and proof nets π with their interpretations $A^{\mathbb{C}}$ and $\pi^{\mathbb{C}}$).

$$
\begin{tikzcd}
 & & A(b,b) \otimes A^{\perp}(b,b) \arrow[dr, "\hat{\bot}_{A(b,b)}"] & \\
\Gamma_1^{\perp} \otimes \Gamma_2^{\perp} \arrow[urr, bend left, "(\pi_1')_b \otimes \pi_2'"] \arrow[r, "\pi_1 \otimes \pi_2'"'] \arrow[drr, bend right, "\pi_1 \otimes \pi_2"'] & \int_x A(x,x) \otimes A^{\perp}(b,b) \arrow[ur, "\delta_b^A \otimes A^{\perp}"] \arrow[dr, "\int_x A \otimes (\delta_b^A)^{\perp}"] & & \bot_{\mathbb{C}} \\
 & & \int_x A(x,x) \otimes \int^y A^{\perp}(y,y) \arrow[ur, "\hat{\bot}_{\int_x A}"'] &
\end{tikzcd}
$$

$\square$

Any map $\phi : \mathtt{Var} \to Ob_{\mathbb{C}}$ extends into a map $\varphi : \mathscr{L}^2 \to Ob_{\mathbb{C}}$ by letting $(A \otimes B)^{\varphi} = A^{\varphi} \otimes B^{\varphi}$, $(\forall XA)^{\varphi} = \int_x^{\mathscr{F}} A^{\varphi}(x,x)$ and $(A^{\perp})^{\varphi} = (A^{\varphi})^{\perp}$. The following can be verified by induction on formulas:

Lemma 2. *For each map $\phi : \mathtt{Var} \to Ob_{\mathbb{C}}$ and each sequent Γ, $\Gamma^{\mathbb{C}}(X_1^{\phi}, \ldots, X_n^{\phi}) = \Gamma^{\phi}$.*

By letting $\Phi(\pi) := \pi^{\mathbb{C}}(X_1^{\phi}, \ldots, X_n^{\phi})$, for $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$, we finally get:

Theorem 1 (functor $\Phi : \mathbb{G} \to \mathbb{C}$)**.** *For all $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$ there exists a functor $\Phi : \mathbb{G} \to \mathbb{C}$ such that, for all $A \in \mathbb{L}^2$, $\Phi(A) = A^{\varphi}$.*

To account for multiplicative units we must introduce *extended G-proof structures*, i.e. G-proof structures including two links with no premiss and unique conclusions $\mathbf{1}$ and $\bot$, respectively, and with lax thinning edges (in the sense of [22]) connecting any occurrence of $\bot$ with a node. Extended G-nets are defined with the usual criterion. Cut-elimination extends straightforwardly to extended G-nets. Extended G-nets can be sequentialized into the sequent calculus for MLL2 with units.

The interpretation $\pi^{\mathbb{C}}$ extends in a straightforward way to extended G-nets. When no quantifier appears in an extended G-net π, then this net corresponds to a *lax linking* in the sense of [22], p.22. We will exploit the result contained in [22] that the category Lax of lax linkings modulo rewiring (see section 6) is the free *-autonomous category.

We can now define the equivalence relation generated by the interpretation of G-nets:

Definition 2 (equivalence $\simeq_\varepsilon$). *We let $\simeq_\varepsilon$ be the equivalence relation over G-nets given by $\pi \simeq_\varepsilon \pi'$ iff $\pi^{\mathbb{C},\mathscr{F}} = (\pi')^{\mathbb{C},\mathscr{F}}$, for any dinatural model $\mathscr{F}$ over a *-autonomous category $\mathbb{C}$. We let $\mathbb{G}_\varepsilon$ be the category of cut-free G-nets considered modulo $\simeq_\varepsilon$.*

From proposition 1 it follows that $\simeq_\varepsilon$ includes $\beta\eta$-equivalence (hence it is a congruence). The following example shows that $\simeq_\varepsilon$ strictly extends $\beta\eta$-equivalence. In the next section we will consider a more general example related to the Yoneda isomorphism.

Example 1. *The category $\mathbb{G}$ is not *-autonomous (while $\mathbb{G}_\varepsilon$ is). In particular, $\forall X(X^\perp \parr X)$ is not a tensor unit in $\mathbb{G}$: by composing any G-net in $\mathbb{G}(Y \otimes \forall X(X^\perp \parr X), Y)$ with the unique G-net in $\mathbb{G}(Y, Y \otimes \forall X(X^\perp \parr X))$ one cannot get $id_{Y\otimes\forall X(X^\perp \parr X)}$.*

3 The Yoneda translation

We introduce a way to translate proof nets in (a fragment of) $\mathsf{MLL2}$ into proof nets in MLL which is related to the *Yoneda isomorphism*. The latter is usually stated as a natural bijection $h : \mathsf{Nat}_{\mathbb{C}}(\mathbb{C}(a,x), F(x)) \simeq F(a)$, where $F : \mathbb{C} \to \mathit{Set}$ and $a \in Ob_{\mathbb{C}}$. The maps h and h^{-1} are defined by

$$\begin{array}{rcl} h(\theta_x) = \theta_a(id_a) & & (\theta_x \in \mathsf{Nat}_{\mathbb{C}}(\mathbb{C}(a,x), F(x))) \\ (h^{-1}(z))_x(f) = F(f)(z) & & (z \in F(a), f \in \mathbb{C}(a,x)) \end{array} \tag{3.1}$$

In a dinatural model $\mathscr{F}$, if F, G are covariant functors, $\mathscr{F}(F,G) \simeq \mathscr{F}(\mathbf{1}_{\mathbb{C}}, \int_x^{\mathscr{F}} F(x) \multimap G(x))$ as a consequence of the universality of (relativized) ends and the Yoneda isomorphism can be restated as the isomorphism below:

$$h : \mathscr{F}\left(\mathbf{1}_{\mathbb{C}}, \int_x^{\mathscr{F}} (F \multimap x) \multimap G(x)\right) \simeq \mathscr{F}(\mathbf{1}_{\mathbb{C}}, G \circ F) \tag{3.2}$$

This isomorphism can be expressed in the language of $\mathsf{MLL2}$ by equivalences of the form $\forall X((C \multimap X) \multimap D[X]) \simeq D[C/X]$, where $D[X]$ is a formula in which X occurs only positively. This leads to the following definition:

Definition 3 (Yoneda formula). *Given a variable $X \in \mathtt{Var}$ and a formula $A \in \mathscr{L}^2$, A is* Yoneda in X *(resp.* co-Yoneda in X*) if A (resp. $A^\perp$) is of the form $(\bigotimes_i^n C_i \otimes X^\perp) \parr D[X]$[4], where X does not occur in any of the C_i and $D[X]$ has a unique, positive, occurrence of X.*

We let $\mathscr{L}^2_{\mathscr{Y}} \subset \mathscr{L}^2$ be the language obtained by restricting $\forall$ quantification (resp. $\exists$ quantification) to Yoneda (resp. co-Yoneda) formulas. In other words $\forall XA \in \mathscr{L}^2_{\mathscr{Y}}$ (resp. $\exists XA \in \mathscr{L}^2_{\mathscr{Y}}$) only if $A \in \mathscr{L}^2_{\mathscr{Y}}$ and A is Yoneda in X (resp. co-Yoneda in X). We indicate by $\mathsf{MLL2}_{\mathscr{Y}}$ the restriction of G-nets to $\mathscr{L}^2_{\mathscr{Y}}$.

The Yoneda isomorphism induces a translation from $\mathsf{MLL2}_{\mathscr{Y}}$ formulas into propositional formulas: the *Yoneda translation* $A_{\mathscr{Y}}$ of a formula $A \in \mathscr{L}^2_{\mathscr{Y}}$ is the multiplicative formula obtained by replacing systematically $\forall X((\bigotimes_i^n C_i \otimes X^\perp) \parr D[X])$ by $D[\bigotimes_i^n C_i \otimes \mathbf{1}]$ and $\exists X((\parr_i^n C_i \parr X) \otimes D[X^\perp])$ by $D[\parr_i^n C_i \parr \perp]$. The formulas $\forall X(X^\perp \parr X)$ and $\exists X(X \otimes X^\perp)$ translate the multiplicative units $\mathbf{1}, \perp$. We let $\mathscr{L}_{\mathbf{1},\perp} \subset \mathscr{L}^2_{\mathscr{Y}}$ be the language obtained by restricting $\forall XA$ to $A = X^\perp \parr X$ and $\exists XA$ to $A = X \otimes X^\perp$. We let $\mathsf{MLL2}_{\mathbf{1},\perp}$ be the restriction of G-nets to $\mathscr{L}_{\mathbf{1},\perp}$.

Let us fix a dinatural model $\mathscr{F}$ over a $\mathbb{C}$. For any formula A Yoneda in X, the isomorphism 3.2 takes the form $h_A : (\forall XA)^{\mathbb{C},\mathscr{F}} \to A_{\mathscr{Y}}^{\mathbb{C},\mathscr{F}}$[5]. h_A can be represented by means of the extended G-nets $Yo_1^A \in$

[4]Given a formula A and a finite (possibly empty) sequence of formulas $C_1, \dots, C_n$, we indicate by $\bigotimes_i^n C_i \otimes A$ (resp. $\parr_i^n C_i \parr A$) the formula $C_1 \otimes \dots \otimes C_n \otimes A$ (resp. $C_1 \parr \dots \parr C_n \parr A$).

[5]It is easily seen that the Yoneda isomorphism can be restated for relativized coends in a dinatural model.

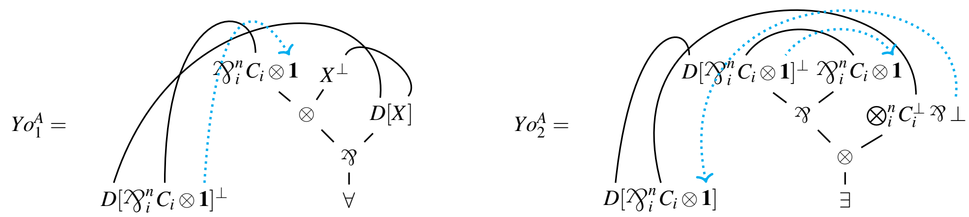


Figure 2: G-nets for the Yoneda isomorphism

$\mathbb{G}(\forall XA, A_{\mathscr{Y}})$ and $Yo_2^A \in \mathbb{G}(A_{\mathscr{Y}}, \forall XA)$ illustrated in figure 2 (where the blue arrows correspond to lax thinning edges). By inspecting the behavior of these G-nets with respect to cut-elimination one easily sees that they correspond to h_A in the following sense:

Lemma 3 (Yoneda isomorphism for G-nets)**.** *Let A be Yoneda in X,*

1. *For all G-net π of conclusion $\forall XA$, $(Yo_1^A \circ \pi)^{\mathbb{C},\mathscr{F}} = h_A(\pi^{\mathbb{C},\mathscr{F}})$.*
2. *For all G-net π of conclusion $\exists XA^\perp$, $(Yo_2^A \circ \pi)^{\mathbb{C},\mathscr{F}} = h_A^{-1}(\pi^{\mathbb{C},\mathscr{F}})$.*

Let $\mathbb{G}^{\mathscr{Y}}$ (resp. $\mathbb{G}_{\varepsilon}^{\mathscr{Y}}$) be the subcategory of $\mathbb{G}$ made of G-nets (resp. G-nets modulo $\simeq_\varepsilon$) in the fragment $\mathsf{MLL2}_{\mathscr{Y}}$. By using the extended G-nets Yo_1^A, Yo_2^A, the Yoneda translation can be extended into a functor $\mathbf{Yon} : \mathbb{G}^{\mathscr{Y}} \to \mathsf{Lax}$, where Lax is the category of lax linkings for MLL recalled in the previous section. The functor **Yon** associates to a $\mathscr{L}^2_{\mathscr{Y}}$ formula A its translation $A_{\mathscr{Y}}$ and to a G-net π of conclusions Γ the lax linking $\mathbf{Yon}(\pi)$ of conclusions $\Gamma_{\mathscr{Y}}$ obtained by cutting any occurrence of $\forall XA$ (resp. $\exists XA^\perp$) in π with Yo_1^A (resp. with Yo_2^A).

More precisely, $\pi_{\mathscr{Y}}$ is constructed as follows: since π is sequentializable, for any $\exists$-link of conclusion $\exists XA$, there exists a sub-net π_A of conclusions $\Gamma, A[B/X]$ from which π can be obtained by first adding the $\exists$-link and then adding other links. Starting from the topmost $\exists$-links in the sequentialization of π, let us replace the associated sub-nets π_A with the sub-net π_A^* obtained by cutting π_A with Yo_A^1 and then reducing this cut. After eliminating all $\exists$-links, the same construction, with Yo_A^2 in place of Yo_A^1 allows to eliminate $\forall$-links. $\pi_{\mathscr{Y}}$ is clearly independent from the sequentialization chosen. However, by reasoning by induction on the sequentialization order one can be convinced that all cuts so introduced can be eliminated. A simple verification also shows that the transformation just defined is functorial (i.e. it preserves identity and composition).

As a functor from $\mathbb{G}^{\mathscr{Y}}$ to Lax, **Yon** is not faithful: for instance, the composition $Yo_1^A \circ Yo_2^A$ is not equal to the identity on $\forall XA$, while its translation yields the identity on $A_{\mathscr{Y}}$. This implies that the G-net representation of the Yoneda isomorphism is not an isomorphism in $\mathbb{G}^{\mathscr{Y}}$. This is another way to say that the equivalence $\simeq_\varepsilon$ strictly extends $\beta\eta$-equivalence of G-nets.

However, the Yoneda isomorphism becomes an isomorphism of G-nets as soon as we consider these modulo $\simeq_\varepsilon$. More generally, by applying the "true" Yoneda isomorphism as well as lemma 3, we obtain the following:

Lemma 4. **Yon** *is faithful as a functor from $\mathbb{G}_{\varepsilon}^{\mathscr{Y}}$ to* Lax.

In the next section we will introduce a compact representation of G-nets which allows to compute the equivalence $\simeq_\varepsilon$ in a syntactic way.

4 Linkings for $\mathsf{MLL2}_{\mathscr{Y}}$

In this section we introduce a compact representation of proof nets for $\mathsf{MLL2}_{\mathscr{Y}}$. We adopt a notion of *linking* inspired from [22, 21] and a notion of *rewiring* inspired from [6, 18, 22] (in which the role of thinning edges is given by *witness edges*). In particular, the restriction to $\mathscr{L}^2_{\mathbf{1},\perp}$ yields a formalism which is equivalent to lax linkings for MLL (lemma 8).

Given a formula A (resp. a sequent Γ) we let $tA = (nA, eA)$ (resp. $t\Gamma = (n\Gamma, e\Gamma)$) be its parse tree (resp. parse forest). We will often confuse the nodes of Γ with the associated formulas. Let Γ be a clean sequent. An *edge* e is a pair of leaves of $t\Gamma$ consisting in two occurrences of opposite polarity of the same variables. Any $\exists$-link in $t\Gamma$ has a distinguished eigenvariable. A variable is an *existential variable* if it occurs quantified existentially. We will indicate existential variables as $\mathbf{X}, \mathbf{Y}, \dots$, to stress that these variables are treated as "unknown variables". A formula containing no free occurrences of existential variables will be called a *ground formula*. Since in all formulas of the form $\exists XA$, A is co-Yoneda in X, existential variables come in pairs, called *co-edges*. We let $\Gamma^{\exists}$ be the set of co-edges of Γ. Any co-edge c is uniquely associated with an existential formula A_c. For any formula B and co-edge c, we say that B *depends on* c when $c = (\mathbf{X}, \mathbf{X}^{\perp})$ and $\mathbf{X}$ occurs free in B.

A *linking* of Γ is a set of disjoint edges whose union contains all but the existential variables of Γ. A *witnessing function* over Γ is an injective function $W : \Gamma^{\exists} \to n\Gamma$, associating any co-edge with a node of Γ. We will represent witnessing functions by using colored and dotted arrows, called *witness edges*, going from the two nodes of a co-edge c to the formula $W(c)$. An $\exists$-*linking* over Γ is a pair $\ell = (E, W)$, where E is a linking over Γ and W is a witnessing function over Γ. Examples of $\exists$-linkings are shown in fig. 3c.

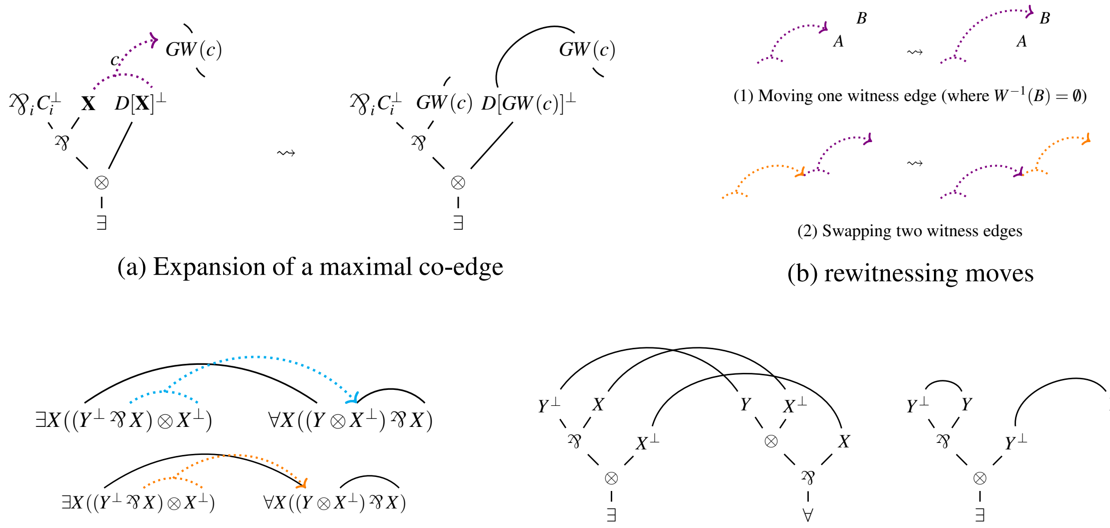
Given a witnessing function W, we let the *dependency graph of* W be the directed graph D_W with nodes the co-edges and arrows $c \to c'$ when $W(c)$ depends on c'. We call a witnessing function W *acyclic* when the graph D_W is directed acyclic. We call $\ell = (E, W)$ *acyclic* when W is acyclic. When D_W is acyclic, the witnessing function W allows to associate a ground formula (called a *ground witness*) $GW(c)$ to any co-edge: if c is a leaf of D_W, then $W(c)$ is a already ground formula, so $GW(c) := W(c)$; otherwise, if D_W contains the edges $(c, c_1), \dots, (c, c_n)$, $W(c)$ depends on the existential variables $\mathbf{X}_1, \dots, \mathbf{X}_n$ associated to the co-edges $c_1, \dots, c_n$, respectively, then by induction on the well-founded order induced by D_W, we can suppose the $GW(c_i)$ well-defined and put $GW(c) := W(c)[GW(c_1)/\mathbf{X}_1, \dots, GW(c_n)/\mathbf{X}_n]$.

Acyclic $\exists$-linkings provide a compact representation of G-proof structures, since to an $\exists$-linking $\ell = (E, W)$ can be associated a unique G-proof structure $\pi(\ell)$ as follows: starting from co-edges which are leaves in D_W, we repeatedly apply to the graph $E \cup t\Gamma$, recursively on D_W, the *co-edge expansion* operation shown in fig. 3a, which instantiates the unknown variable of a co-edge c with its ground witness $GW(c)$. An $\exists$-linking ℓ is *correct* when it is acyclic and $\pi(\ell)$ is a G-net.

We introduce an equivalence relation over correct $\exists$-linkings, called *rewitnessing*, inspired from the "rewiring" technique in [6, 18, 22]. Given a witnessing function W, a *simple rewitnessing of* W is a witnessing function W' obtained by either moving exactly one witness edge from one formula to another "free" one (i.e. to some formula A such that $W^{-1}(A) = \emptyset$), or by switching two consecutive witness edges, i.e. two edges c_1, c_2 such that $W(c_1) \in c_2$, as shown in fig. 3b. We let $\ell \sim_1 \ell'$ if $\ell = (E, W)$, $\ell' = (E, W')$ and W' is a simple rewitnessing of W. We let $\sim$ be the reflexive and transitive closure of $\sim_1$.

In fig. 3c are shown $\sim$-equivalent $\exists$-linkings over $\exists X((Y^{\perp} \parr X) \otimes X^{\perp}), \forall X((Y \otimes X^{\perp}) \parr X)$. These correspond to the two $\simeq_{\varepsilon}$-equivalent G-nets in fig. 3d. In the next section we will show that rewitnessing can be used to compute the ε-equivalence. When A is Yoneda in X, we let $ID_{\forall XA}$ denote the $\exists$-linking in figure 7a.

We let $\mathbb{L}^{\exists}$ be the *category of* $\exists$-*linkings*, whose objects are the formulas of $\mathsf{MLL2}_{\mathscr{Y}}$ and where



(a) Expansion of a maximal co-edge

(b) rewitnessing moves

(c) $\sim$-equivalent $\exists$-linkings

(d) $\simeq_\varepsilon$-equivalent G-nets

Figure 3: $\exists$-linkings and rewitnessing.

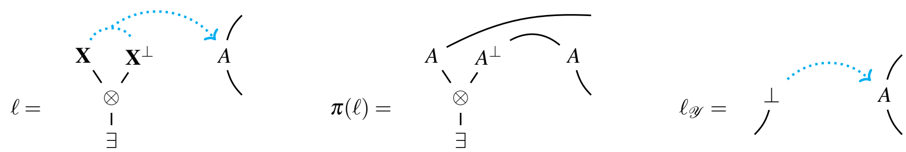


Figure 4: Local comparison of ℓ, $\pi(\ell)$ and $\ell_{\mathscr{Y}}$ for $\perp^{\exists} = \exists X(X \otimes X^{\perp})$.

$\mathbb{L}^{\exists}(A,B)$ is the set of $\sim$-equivalence classes of correct $\exists$-linkings of conclusions $A^{\perp}, B$, with composition given by cut-elimination (see next section). We let $\mathbb{L}^{\mathbf{1},\perp}$ be the restriction of $\mathbb{L}^{\exists}$ to $\mathsf{MLL2}_{\mathbf{1},\perp}$ formulas.

Similarly to the functor $\mathbf{Yon} : \mathbb{G} \to \mathsf{Lax}$, we can construct a functor $\mathscr{Y} : \mathbb{L}^{\exists} \to \mathit{Lax}$ for $\exists$-linkings. The linking $\ell_{\mathscr{Y}}$ is obtained in two steps: first, for any co-edge $c = (\mathbf{X}, \mathbf{X}^{\perp})$, replace A_c by $(A_c)_{\mathscr{Y}}$, replace the thinning edge from c to $W(c)$ by a lax thinning edge from $\perp$ to $W(c)$, and move all lax thinning edges pointing to $\mathbf{X}$ or $\mathbf{X}^{\perp}$ (or to $\mathbf{X} \otimes \mathbf{X}^{\perp}$ if $A_c = \perp^{\exists}$) onto $W(c)$; once all co-edges have been eliminated, replace any universal formula $\forall XA$ by $(\forall XA)_{\mathscr{Y}}$ and eliminate the unique edge $(X^{\perp}, X)$. The transformation just described yields then a lax linking $E_{\mathscr{Y}}$ over the MLL sequent $\Gamma_{\mathscr{Y}}$. Observe that witness edges are replaced by lax thinning edges, see fig. 4.

By letting $\sim_{lax}$ denote the rewitnessing equivalence over lax linkings, we have:

Lemma 5. $\ell \sim \ell' \Rightarrow \ell_{\mathscr{Y}} \sim_{lax} \ell'_{\mathscr{Y}}$.

Proof. The claim follows from the fact that a rewitnessing move of type (1) (fig. 3b) in ℓ corresponds to a rewiring move in $\ell_{\mathscr{Y}}$, while a rewitnessing move of type (2) in ℓ does not affect $\ell_{\mathscr{Y}}$.

□

5 Cut-elimination for $\exists$-linkings

We let a *cut sequent* be a sequent of the form $\Gamma, [\Delta]$, where Γ, Δ is a clean sequent and Δ is a multiset of formulas, called *cut formulas*, of the form $A \otimes A^{\perp}$ (that we depict by a configuration of the form

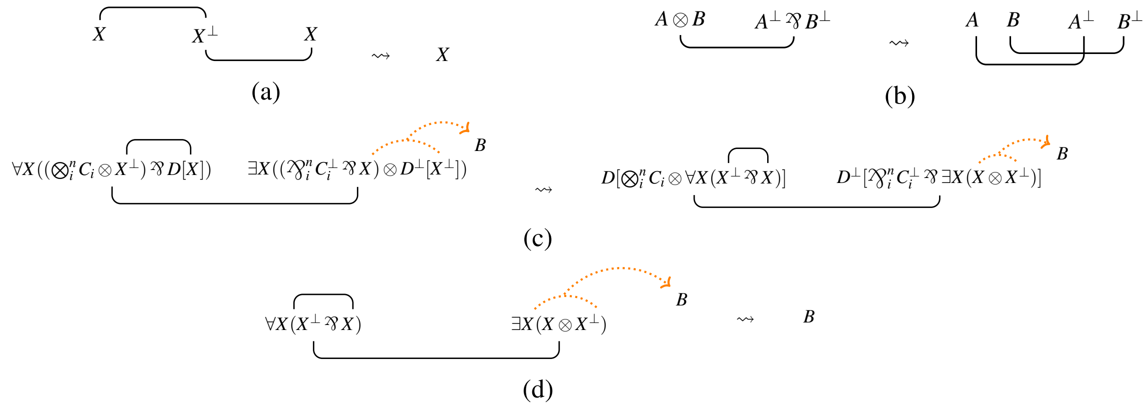


Figure 5: Cut elimination local steps.

A $A^\perp$).

By an $\exists$-linking over $\Gamma, [\Delta]$ we indicate an $\exists$-linking over Γ, Δ. We call an $\exists$-linking $\ell = (E, W)$ *ready* when $W^{-1}(A) = \emptyset$ for all A occurring in a cut-formula. Cut-elimination relies on the following lemma, proved in appendix C.

Lemma 6 ("ready lemma")**.** *For any correct $\exists$-linking ℓ there exists a ready ℓ' such that $\ell' \sim \ell$.*

Indeed, by lemma 6 it suffices to apply cut-elimination to ready $\exists$-linkings. *Cut reduction* is the relation over ready $\exists$-linkings defined by the rewrite rules in figure 5, where in case 5c either $n \geq 1$ or $D[X] \neq X$, and, in case 5c and 5d the existence of the lefthand edge is forced by the fact that Γ, Δ is clean. Observe that the reduction (c) incorporates the Yoneda translation.

We now verify usual properties of cut-elimination.

Lemma 7 (confluence)**.** *Cut reduction is confluent.*

Proof. Immediate consequence of the locality of the reduction rules. □

Proposition 2 (stability)**.** *Let ℓ be a correct and ready. If $\ell \rightsquigarrow \ell'$, then ℓ' is correct.*

Proof. For any G-net π and for any formula $\forall XA$ (with dual formula $\exists XA^\perp$) occurring in a cut, let π^A be the G-net obtained by replacing the formula $\forall XA$ (resp. $\exists XA^\perp$) by $A_{\mathscr{Y}}$ (resp. $A^\perp_{\mathscr{Y}}$) by cutting it with Yo_1^A (resp. Yo_2^A). In other words, we apply the Yoneda translation locally. π^A is still a G-net, as π, Yo_1^A and Yo_2^A are all sequentializable, and the cut introduced can be applied just after the rules introducing the quantifier of $\forall XA$ (resp. $\exists XA^\perp$).

Now, any cut reduction rule $\ell \mapsto \ell'$ induces a transformation of G-nets $\pi(\ell) \mapsto^* \pi(\ell')$. We must show then that $\mapsto^*$ preserves correctness. This is trivial in cases 5a, 5b and 5d. In case 5c, let the cut-formula be $\forall XA \otimes \exists XA^\perp$; then $\pi(\ell) \mapsto \pi^*$, where π^* can be obtained from π^A (which is a G-net as $\pi(\ell)$ is a G-net and G-net reduction preserves correctness) by performing some G-net reduction steps. We conclude then that π^* is correct, i.e. ℓ' is correct.

□

Strong normalization can be proved in a direct way, without reducibility candidates techniques.

Proposition 3 (strong normalization)**.** *Let ℓ be a correct and ready $\exists$-linking over $\Gamma, [\Delta]$. Then all cut-reductions of ℓ terminate over a unique correct $\exists$-linking $nf(\ell)$ over Γ, called the* normal form of ℓ.

Proof. We define a measure $s(A)$ over formulas as follows: $s(X) = s(X^\perp) = 0$, $s(A \otimes B) = s(A \parr B) = s(A) + s(B) + 1$, $s(\forall X(X^\perp \parr X)) = s(\exists X(X \otimes X^\perp) = 1$ and, when either $n \geq 1$ or $D[X] \neq X$, $s(\forall X((\bigotimes_i^n C_i \otimes X^\perp) \parr D[X])) = s(\exists X((\bigparr_i^n C_i^\perp \parr X) \otimes D[X]^\perp)) = s(D[C]) + 3$, where C is either $\bigotimes_i^n C_i$ or $\bigparr_i^n C_i^\perp$. By letting $s(\ell)$ be the sum all $s(A)$, where A is a cut-formula, any reduction step makes $s(\ell)$ decrease strictly. $\square$

By proposition 3 any correct $\exists$-linking has a unique normal form, up to rewitnessing.

6 Characterization of ε-equivalence

We exploit the Yoneda translation to prove that the compact representation of G-nets by means of $\exists$-linkings characterizes the equivalence induced by ends and coends. We will indeed show that the translation $\ell \to \pi(\ell)$ yields an isomorphism of categories $\mathbb{L}^\exists \simeq \mathbb{G}_\varepsilon^{\mathscr{Y}}$.

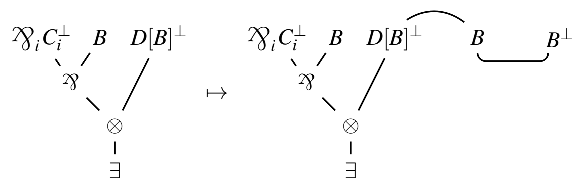


Figure 6: From π to π^{cut}.

We start by defining the translation $\ell : \pi \mapsto \ell_\pi$ "adjoint" to $\pi : \ell \mapsto \pi(\ell)$. First, for a G-net π, let π^{cut} be obtained from π by introducing a new cut for any $\exists$-link of π as follows: if $A_c = \exists X((\bigotimes_i^n C_i \parr X) \bigotimes D[X]^\perp)$ with premiss $(\bigotimes_i^n C_i \parr B) \bigotimes D[B]^\perp$, introduce an axiom and a cut over B as illustrated in fig. 6. By inspecting the co-edge expansion in fig. 3a, it can be seen that π^{cut} is of the form $\pi(\ell^{cut})$ for a unique $\exists$-linking with cuts ℓ^{cut}. We let then ℓ_π be the normal form of ℓ^{cut}. While $\ell = \ell_{\pi(\ell)}$ holds by construction, the converse equation $\pi = \pi(\ell_\pi)$ does not hold in general (since cut-elimination of $\exists$-linking might require rewitnessings). However, we will show that the weaker $\pi \simeq_\varepsilon \pi(\ell_\pi)$ holds (theorem 2).

We can use the translations π and ℓ to relate the Yoneda translations for G-nets and $\exists$-linkings as follows:

Proposition 4. *a.* $\mathbf{Yon} \circ \pi \ = \ \mathscr{Y}$.
b. $\mathscr{Y} \circ \ell \ = \ \mathbf{Yon}$.

$$\begin{tikzcd} \mathbb{G}_\varepsilon^{\mathscr{Y}} \arrow[rr, bend right, "\ell"'] \arrow[dr, "\mathbf{Yon}"'] & & \mathbb{L}^\exists \arrow[ll, bend right, "\pi"'] \arrow[dl, "\mathscr{Y}"] \\ & Lax & \end{tikzcd}$$

Proof. *a.* can be verified by inspecting the reduction steps involved in the transformation of $\pi(\ell)$ into a lax linking. For *b.* we argue as follows: π is β-equivalent to $\pi^{cut} = \pi(\ell^{cut})$, where $\ell^{cut} \sim \ell_\pi$. Now, from *a.* it follows that $\mathbf{Yon}(\pi) = \mathbf{Yon}(\pi^{cut}) = \mathbf{Yon}(\pi(\ell^{cut})) \sim_{lax} \ell_{\mathscr{Y}}^{cut}$. From $\ell_\pi \sim \ell^{cut}$ we deduce then, by lemma 5, that $(\ell_\pi)_{\mathscr{Y}} \sim_{lax} \ell_{\mathscr{Y}}^{cut}$, hence we conclude $(\ell_\pi)_{\mathscr{Y}} \sim_{lax} \mathbf{Yon}(\pi)$. $\square$

From proposition 4 we deduce that if ℓ is correct, $\ell_{\mathscr{Y}}$ is correct (since $\ell_{\mathscr{Y}} = \mathbf{Yon}(\pi(\ell))$). Moreover, we deduce that the functor $\mathscr{Y}$ is faithful (as **Yon** is).

The following proposition allows to state that ℓ is indeed a functor $\ell : \mathbb{G}_\varepsilon^{\mathscr{Y}} \to \mathbb{L}^\exists$.

Proposition 5. *If $\pi \simeq_\varepsilon \pi'$, then $\ell_\pi \sim \ell_{\pi'}$.*

Proposition 5 is deduced from the two lemmas below.

Lemma 8. $\mathbb{L}^\exists$ *is *-autonomous.* $\mathbb{L}^{\mathbf{1},\perp}$ *is the free *-autonomous category.*

Proof. That $\mathbb{L}^\exists$, with units $\forall X(X^\perp \parr X)$ and $\exists X(X \otimes X^\perp)$, verifies all coherence conditions of a *-autonomous category is a simple verification. The second point follows from the faithfulness of $\mathscr{Y}$ and the fact that Lax is the free *-autonomous category ([22]). $\square$

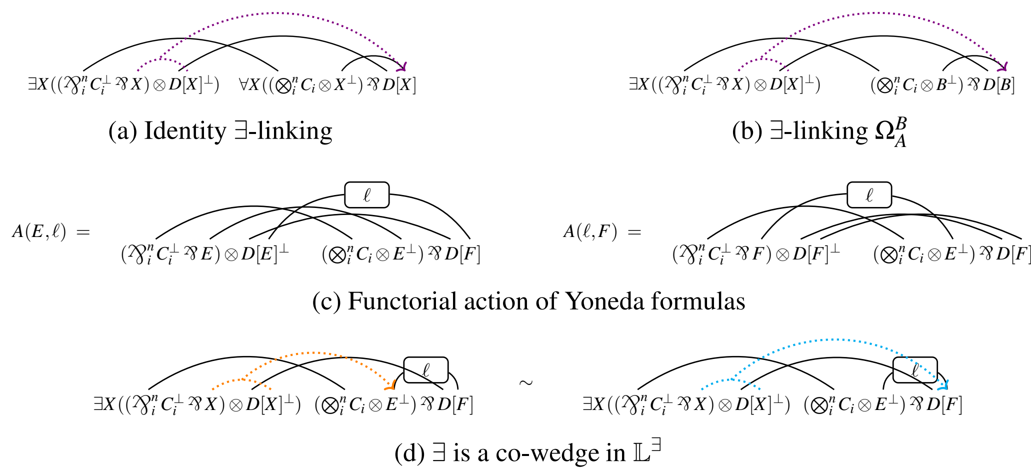


Figure 7: Existential linkings and co-wedges.

For any $A = (\bigparr_i C_i \parr X) \otimes D[X^\perp]$ Yoneda in X and any $B \in \mathscr{L}^2_{\mathscr{Y}}$, let Ω_A^B be the correct $\exists$-linking in fig. 7b. Moreover, for all $B, C \in \mathscr{L}^2_{\mathscr{Y}}$, we let $A(E,\ell)$ and $A(\ell,F)$ be the correct $\exists$-linking in fig. 7c, corresponding to the covariant and contravariant functorial action of A on ℓ. The following lemma states then that the existential quantifier behaves like a co-wedge in $\mathbb{L}^\exists$.

Lemma 9. *For all A Yoneda in X, $E, F \in \mathscr{L}^2_{\mathscr{Y}}$ and $\ell \in \mathbb{L}^\exists(E,F)$, $\Omega_A^E \circ A(\ell,E) \sim \Omega_A^F \circ A(F,\ell)$*

Proof. Indeed $\Omega_A^E \circ A(\ell,E)$ and $\Omega_A^F \circ A(F,\ell)$ differ by a unique rewitnessing, see fig. 7d. ☐

Example 2. *The "Yoneda isomorphism" holds in $\mathbb{L}^\exists$, as the composition $\ell_{Yo_1^A} \circ \ell_{Yo_2^A}$ reduces to $ID_{\forall XA}$ (up to rewitnessing).*

By relying on the two Yoneda translations we now prove our main result.

Theorem 2. *π and ℓ define an isomorphism of categories $\mathbb{G}_\varepsilon^{\mathscr{Y}} \simeq \mathbb{L}^\exists$.*

Proof. We will show that π and ℓ are faithful functors inverse each other. To prove that π is a faithful functor we must show that the assignment $\ell \mapsto \pi(\ell)$ yields an injective function $\mathbb{L}^\exists(A,B) \to \mathbb{G}_\varepsilon^{\mathscr{Y}}(A,B)$. We claim that $\ell \sim \ell' \Rightarrow \pi(\ell) \simeq_\varepsilon \pi(\ell')$: from $\ell \sim \ell'$ we deduce by lemma 5 $\ell_{\mathscr{Y}} \sim_{lax} \ell'_{\mathscr{Y}}$, hence, by proposition 4 *a.*, $\mathbf{Yon}(\pi(\ell)) \sim_{lax} \mathbf{Yon}(\pi(\ell'))$, and from the faithfulness of **Yon** we can conclude $\pi(\ell) \simeq_\varepsilon \pi(\ell')$. This shows that π is a function. Functoriality can be easily verified (by showing that π maps identity linkings into identity G-nets and that it preserves composition). Injectivity is proved as follows: if $\pi(\ell) \simeq_\varepsilon \pi(\ell')$ then, by proposition 5, $\ell = \ell_{\pi(\ell)} \sim \ell_{\pi(\ell')} = \ell'$.

To prove that ℓ is a faithful functor we must show that the assignment $\pi \mapsto \ell_\pi$ yields an injective function $\mathbb{G}_\varepsilon^{\mathscr{Y}}(A,B) \to \mathbb{L}^\exists(A,B)$. The functionality of ℓ follows from proposition 5. By construction it can be verified that the functor ℓ translates an identity G-net into an identity $\exists$-linking and that it preserves composition. Injectivity is proved as follows: if $\ell_\pi \sim \ell_{\pi'}$, then by lemma 5, $(\ell_\pi)_{\mathscr{Y}} \sim_{lax} (\ell_{\pi'})_{\mathscr{Y}}$, hence by proposition 4 *b.*, $\mathbf{Yon}(\pi) \sim_{lax} \mathbf{Yon}(\pi')$ and from the faithfulness of **Yon** we conclude $\pi \simeq_\varepsilon \pi'$.

Since $\ell = \ell_{\pi(\ell)}$, it remains to show that $\pi \simeq_\varepsilon \pi(\ell_\pi)$. This follows from $\ell_\pi = \ell_{\pi(\ell_\pi)}$ and the faithfulness of ℓ.

☐

Corollary 3. *For all G-nets π, π' of conclusions Γ, $\pi \simeq_\varepsilon \pi'$ iff $\ell_\pi \sim \ell_{\pi'}$.*

7 Conclusions

We provided a syntactic characterisation of the equational theory generated by ends/coends over *Yoneda* formulas in MLL2. Our result relies on the simple structure of *Yoneda* formulas (1 positive and 1 negative occurrence of quantified variables) and on the existence of a faithful translation from $\mathsf{MLL2}_{\mathscr{Y}}$ to MLL with units. It seems thus plausible that more sophisticated syntactic techniques are required to extend the characterisation to more expressive fragments of MLL2. In particular, while our result implies the decidability of the dinatural equivalence $\simeq_\varepsilon$ in $\mathsf{MLL2}_{\mathscr{Y}}$, it is not known whether the theory $\simeq_\varepsilon$ is decidable over full MLL2. However, keeping the Yoneda restriction, it can be expected that similar characterizations can be obtained for more expressive systems like MELL2 (which is as expressive as System F).

Finally, it might be interesting to compare the theory $\simeq_\varepsilon$ with the equivalence arising from other models of MLL2 investigated in the literature. For instance, while it is well-known that the coherent model of second order linear logic [13] is not dinatural ([11]), it can be easily seen that it satisfies the Yoneda isomorphism. Hence it can be conjectured that the model is injective (in the sense of [7]) with respect to $\exists$-linkings for $\mathsf{MLL2}_{\mathscr{Y}}$.

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A *-autonomous categories and coends

We recall that a *-autonomous category is a category $\mathbb{C}$ endowed with functors $\_ \otimes \_ : \mathbb{C}^2 \to \mathbb{C}$ and $\_^{\perp} : \mathbb{C}^{op} \to \mathbb{C}$, an object $\mathbf{1}_{\mathbb{C}}$, the following natural isomorphisms:

$$
\begin{aligned}
\alpha_{a,b,c} &: a \otimes (b \otimes c) \to (a \otimes b) \otimes c \\
\lambda_a &: a \otimes \mathbf{1}_{\mathbb{C}} \to a \\
\rho_a &: \mathbf{1}_{\mathbb{C}} \otimes a \to a \\
\sigma_{a,b} &: a \otimes b \to b \otimes a
\end{aligned}
$$

and a natural bijection between $\mathbb{C}(a \otimes b, c)$ and $\mathbb{C}(a, b^{\perp} \parr c)$, where $x \parr y = \mathbb{C}(x^{\perp}, y)$, satisfying certain coherence conditions (that we omit here, see [2]). In any *-autonomous category $\mathbb{C}$ there is a natural isomorphism $A^{\perp\perp} \simeq A$. $\mathbb{C}$ is called *strict* when this isomorphism is an identity.

For the definition of multivariant functors and dinatural transformations the reader can look at [28]. When $F : (\mathbb{C}^{op} \otimes \mathbb{C})^{n+1} \to \mathbb{D}$ and the values $a_1, \dots, a_n \in Ob_{\mathbb{C}}$ are clear from the context, we will will often abbreviate $F((a_1, \dots, a_n, a), (a_1, \dots, a_n, b))$ as $F(a, b)$.

Given $\mathbb{C}$ *-autonomous, for all $a \in Ob_{\mathbb{C}}$, there exist dinatural transformations $\hat{\mathbf{1}}_x : \mathbf{1}_{\mathbb{C}} \to x^{\perp} \parr x$ and $\hat{\perp}_x = \hat{\mathbf{1}}_x^{\perp} : x \otimes x^{\perp} \to \perp_{\mathbb{C}}$, where $\perp_{\mathbb{C}} := \mathbf{1}_{\mathbb{C}}^{\perp}$. It is clear that such transformations exist in all dinatural model, according to definition 1.

Given categories $\mathbb{C}, \mathbb{D}$ and a multivariant functor $F : (\mathbb{C}^{op} \otimes \mathbb{C})^{n+1} \to \mathbb{D}$, a *wedge for F*[6] (dually, a *co-wedge for F*, see [28]) is a pair $(C, \delta_{x_1,\dots,x_n,a})$ (resp. $(D, \omega_{x_1,\dots,x_n,a})$)[7] made of a functor $C : (\mathbb{C}^{op} \otimes \mathbb{C})^n \to \mathbb{D}$ and a dinatural transformation $\delta_a : C \to F(a,a)$ (resp. $\omega_a : F(a,a) \to D$) natural in $x_1, \dots, x_n$. A wedge (resp. a co-wedge) for F is an *end* (resp. a *coend*) when the dinatural transformation δ_a (resp. ω_a) is *universal*. This means that for any functor $G : (\mathbb{C}^{op} \otimes \mathbb{C})^n \to \mathbb{D}$ and dinatural transformation $\theta_a : G \to F(a,a)$ (resp. $\theta_a : F(a,a) \to G$) there exists a unique natural transformation $h : G \to \int_x F(x,x)$ (resp. $k : \int^x F(x,x) \to G$) such that the following diagrams commute for all $f \in \mathbb{C}(a,b)$:

[6] We give here a functorial definition of ends and coends which can be easily deduced from the usual definition (see [28]).

[7] We will abbreviate $\delta_{x_1,\dots,x_n,a}$ and $\omega_{x_1,\dots,x_n,a}$ simply as δ_a and ω_a, respectively.

$$
\begin{tikzcd}
G \arrow[drr, bend left, "\theta_a"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "\theta_b"'] & & \\
& \int_x F(a,a) \arrow[r, "\delta_a"] \arrow[d, "\delta_b"] & F(a,a) \arrow[d, "F(a,f)"] \\
& F(b,b) \arrow[r, "F(f,b)"'] & F(a,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(b,a) \arrow[r, "F(f,a)"] \arrow[d, "F(b,f)"] & F(a,a) \arrow[d, "\omega_a"] \arrow[ddr, bend left, "\theta_a"] & \\
F(b,b) \arrow[r, "\omega_b"] \arrow[drr, bend right, "\theta_b"'] & \int^x F(x,x) \arrow[dr, dashed, "k"] & \\
& & G
\end{tikzcd}
$$

Duality yields $\int_x F = (\int^x F^\perp)^\perp$, $\int^x F = (\int_x F^\perp)^\perp$ and $\delta_a = \omega_a^\perp$, $\omega_a = \delta_a^\perp$.

We recall some basic facts about coends (see [28, 27]):

- Commutation with $⅋/\otimes$:

$$\int_x (F ⅋ G(x,x)) \simeq G ⅋ \int_x G(x,x) \tag{A.1}$$

$$\int^x (F \otimes G(x,x)) \simeq F \otimes \int^x G(x,x) \tag{A.2}$$

- "Fubini" theorem:

$$\int_x \int_y F \simeq \int_y \int_x F \tag{A.3}$$

$$\int^x \int^y F \simeq \int^y \int^x F \tag{A.4}$$

- Commutation of $\int_x / \int^x$ and $⅋$: given a functor F and a multivariant functor $G(x,y)$, there exist natural transformations

$$\mu : \int_x (F ⅋ G(x,x)) \to F ⅋ \int_x G(x,x) \tag{A.5}$$

$$\nu : \int^x (F ⅋ G(x,x)) \to F ⅋ \int^x G(x,x) \tag{A.6}$$

In a dinatural model (def. 1) one considers *relativized* ends and coends, that is, wedges/co-wedges which are universal with respect to a certain class of (composable) dinatural transformations. All facts above about ends and coends can be straightforwardly adapted to relativized ends and coends.

B Hughes sequentialization theorem

We adapt the sequentialization algorithm for unification nets in [21] to G-nets. This algorithm is based on the translation of a unification net into a MLL^- proof net (where MLL^- indicates MLL without units), called the *frame*, by a suitable encoding of jumps. The reconstruction of a sequent calculus derivation exploits then the usual splitting property of MLL^- proof nets. This construction can be straightforwardly adapted to G-nets, by translating a cut-free G-proof structures into MLL^- proof-structures as follows:
(1) *Encode every jump from a* $\forall$ *to an* $\exists$ *as a new link*: for each such jump between formulas $\forall XA$ and $\exists YB$, let Z be a fresh variable. Replace $\exists YB$ by $Z \otimes \exists YB$ and $\forall XA$ by $Z^\perp ⅋ \forall XA$;
(2) *Delete quantifiers*. After (1) replace every formula $\forall XA$ by A and every formula $\exists XA$, with premiss $A[B/X]$, by $A[B/X]$.
We let π_m, the *frame of* π, be the MLL^- proof-structure obtained. The following two lemmas are as in [21].

Lemma 10. *If π is a G-net, π_m is a proof net.*

Lemma 11. *No $\otimes$ added during the construction of π_m splits.*

We can now use π_m to find splitting tensors in π, yielding the following:

Theorem 4 (sequentialization)**.** *If π is a G-net, then π is the translation of some sequent calculus derivation.*

Proof. The sequentialization algorithm for a G-net π is as follows:

1. Start by eliminating negative links, i.e. $⅋, \forall$ links; in other words, for any link of conclusion $A ⅋ B$ (resp. $\forall XA$), let π' be the G-net obtained by deleting the $⅋$ (resp. $\forall$) link. By induction hypothesis π' is sequentializable, yielding a derivation of $\Gamma - \{A ⅋ B\}, A, B$ (resp. $\Gamma - \{\forall XA\}, A$), from which a derivation of Γ can be obtained by a $⅋$-rule (resp. by a $\forall$-rule - we are here supposing that $\Gamma, \forall XA$ is clean, so X does not occur free in Γ).

2. If, after 1, there are $\exists$-links with no incoming jumps, eliminate them; in other words, for any such link of conclusion $\exists XA$, let π' be the G-net obtained by deleting the link. By induction hypothesis π' is sequentializable, yielding a derivation of $\Gamma - \{\exists XA\}, A[B/X]$, for some formula B, from which a derivation of Γ can be obtained by a $\exists$-rule.

3. After 2 all non-axiom links are either $\otimes$ or $\exists$ with incoming jumps. If there is none we are done. Otherwise π_m has only $\otimes$-links, so one must be splitting, and by lemma 11 it corresponds to a splitting $\otimes$ in π. By deleting this link we obtain two G-nets π_1, π_2 yielding, by induction hypothesis, two derivations of conclusions, respectively, Γ_1, A and Γ_2, B, where $\Gamma = \Gamma_1, \Gamma_2, A \otimes B$. Now, a derivation of Γ is obtained by a $\otimes$-rule.

$\square$

C Proof of lemma 6

To prove lemma 6 (the "ready lemma") we use the following facts, which can be easily established by looking at $\pi(\ell)$:

Lemma 12 ($\perp^{\exists}$-moves)**.** *(i.) If $A_c = \perp^{\exists}$ and $W(c) = B$ occurs in a cut-formula $B \otimes B^{\perp}$, then c can be rewired on $B^{\perp}$.*
(ii.) If $A_c = \perp^{\exists}$ and $W(c) = B$, then c can be rewired on any subformula of B.
(iii.) If $A_c = \perp^{\exists}$ and $W(c) = X$ is the conclusion of an axiom link of conclusions $X, X^{\perp}$, then c can be rewired on $X^{\perp}$.

From lemma 12 we deduce:

Proposition 6. *If for all $c \in \Gamma^{\exists}$, $A_c = \perp^{\exists}$, then ℓ is equivalent to a ready $\exists$-linking.*

Proof. For any cut formula $B \otimes B^{\perp}$, there is at least an axiom link going outside the tree of B and $B^{\perp}$, otherwise both B and $B^{\perp}$ would be provable. Hence, if $W(c)$ is in the tree of a cut formula $B \otimes B^{\perp}$, by lemma 12 it can be rewitnessed upwards so to pass through an axiom links moving outside the cut.

$\square$

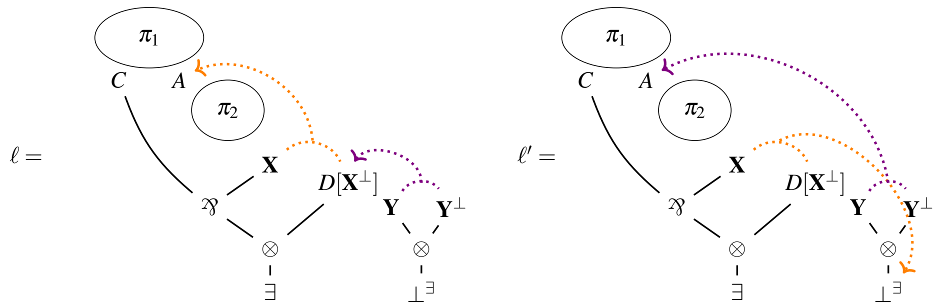

Figure 8: From ℓ to ℓ' by two rewitnessing moves.

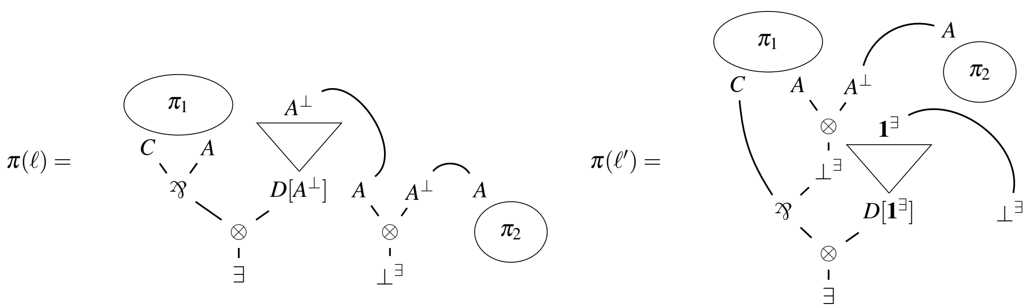

Figure 9: $\pi(\ell)$ and $\pi(\ell')$ are both correct.

Proof of lemma 6. Given $\ell = (E, W)$, we will first construct an $\exists$-linking $\ell^* = (E, W^*)$ such that $\ell \sim \ell^*$ and for all formula A occurring in a cut, $(W^*)^{-1}(A)$ is either empty of contains a formula of the form $\perp^\exists$. From this we can conclude then by applying proposition 12.

Let $c = (\mathbf{X}, \mathbf{X}^\perp) \in \Gamma^\exists$ be such that A_c is not of the form $\perp^\exists$ and $W(c) = A$ occurs in a cut. We can suppose that $W^{-1}(\mathbf{X}^\perp)$ contains $c' = (\mathbf{Y}, \mathbf{Y}')$ such that $A_{c'} = \perp^\exists$ is a conclusion of ℓ and such that $W^{-1}(\perp^\exists) = \emptyset$: if it is not the case then we can add the formula $\perp^\exists$ to the conclusions of ℓ and set $W(c') = \mathbf{X}^\perp$, as this preserves correctness and does not alter equivalence questions because of the isomorphism between the conclusions Γ of ℓ and $\Gamma \parr \perp^\exists$. We let then W' be like W but for $W'(c) = \perp^\exists$ and $W'(c') = A$ (as illustrated in figure 8). W' is obtained from W by a rewitnessing move of type (2) (switching $W(c)$ and $W(c')$ so that c is sent to $\mathbf{Y}$ and c' to A) and a rewitnessing move of type (1) (moving c from $\mathbf{Y}$ to $\perp^\exists$). We must then show that $\ell' = (E, W')$ is correct, so that $\ell \sim \ell'$. This follows by remarking that the first rewitnessing move does not change $\pi(\ell)$ and that the second rewitnessing move transforms $\pi(\ell)$ into $\pi(\ell')$ (as illustrated in fig. 9), preserving correctness, as it can be seen by inspecting paths in both graphs. By applying this operation to all co-edges c such that $A_c \neq \perp^\exists$ we obtain the desired $\exists$-linking $\ell^* \sim \ell$.

$\square$