

Pivoting makes the ZX-calculus complete for real stabilizers

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We show that pivoting property of graph states cannot be derived from the axioms of the ZX-calculus, and that pivoting does not imply local complementation of graph states. Therefore the ZX-calculus augmented with pivoting is strictly weaker than the calculus augmented with the Euler decomposition of the Hadamard gate. We derive an angle-free version of the ZX-calculus and show that it is complete for real stabilizer quantum mechanics.

The ZX-calculus is a formal theory for reasoning about quantum computational systems [3]. It consists of a graphical language based on the Pauli Z and X observables, and a collection of axioms expressed as graph rewrite rules. The ZX-calculus is expressive enough to represent any quantum circuit, and its equations are complete for the stabilizer fragment of quantum mechanics [1]. Due to its graphical nature, and its close relationship to the Z and X observables, the ZX-calculus is particularly well adapted to the study of graph states and measurement-based quantum computation [9, 7].

In addition to the two observables, the ZX-calculus also contains an operator for the Hadamard map: this is the map which exchanges the Z and X bases, and thus provides a duality principle for the graphical language. In previous work [8] the authors showed that if the Hadamard can be expressed in terms of Z and X rotations—that is, as an Euler decomposition—then Van Den Nest’s theorem [14] about local complementation of graph states follows, and vice versa. Furthermore, these results cannot be derived from the original axioms, hence the theory “ZX-calculus + Euler” is strictly stronger than the plain ZX-calculus.

In this paper we find a theory intermediate between the two, albeit having a similar flavour. We consider an operation on graph states called *pivoting* and show that its defining property is equivalent to the possibility to express (one of) the Pauli matrices in terms of the Hadamard. Since pivoting can be done via local complementation, “ZX-calculus + Euler” is stronger than “ZX-calculus + Pivot”. However, we will show that, once again, these equations cannot be derived from the plain ZX-calculus.

The theory “ZX-calculus + Euler” is known to be complete for the stabiliser fragment of quantum mechanics [1]: the stabiliser fragment corresponds to the sub-calculus where all angles are multiples of $\pi/2$. We show that the intermediate calculus “ZX-calculus + Pivot” is complete for the *real* stabiliser fragment of quantum mechanics, and that this fragment admits an angle-free axiomatisation.

Real quantum mechanics is sufficient for quantum computing [2], in the sense that any unitary evolution on n -qubits can be simulated (using a simple encoding) by a real unitary evolution acting on $n + 1$ qubits. As a consequence, while not complete for (complex) quantum mechanics, the intermediate calculus “ZX-calculus + Pivot” might be useful and simpler than the the whole “ZX-calculus + Euler” for proving properties of quantum systems, for example via rewriting.

Remark. There is some variation about which axioms comprise the ZX-calculus. In [8] we considered fewer axioms than we do here; whereas in some later work, notably [1], the Euler decomposition of the Hadamard is included as an axiom.

1 The Graphical Formalism

We recall the syntax, semantics, and basic properties of the ZX-calculus. For a full exposition, see [3].

Definition 1.1. An *open graph* is a triple (G, I, O) consisting of a finite undirected graph $G = (V, E)$ and distinguished subsets $I, O \subseteq V$ of degree one vertices, called the *inputs* and *outputs*, respectively. The set of vertices $I \cup O$ is called the *boundary* of G , and $V \setminus (I \cup O)$ is the *interior* of G . An open graph is called *empty* if its interior is empty; it is called *prime* if it is connected and its interior is a singleton.

We view the inputs and outputs as finite ordinals, and write $\gamma : n \rightarrow m$ for a graph with n inputs and m outputs. Open graphs form a self-dual compact category: composition is achieved by identifying the inputs of one graph with the outputs of another and erasing the resulting vertices; the tensor product is simple juxtaposition of graphs. The unit and counit maps are generated from the unique empty graphs $d : 0 \rightarrow 2$ and $e : 2 \rightarrow 0$. Note that, due to general results [10, 15, 6], a pair of graphs can be deformed from one to other if and only if they are equal by the axioms of compact categories.

The terms of the ZX-calculus are certain open graphs we call *diagrams*.

Definition 1.2. A *diagram* is an arrow of the free category \mathcal{D} generated by the following prime graphs:

$$Z_m^n(\alpha) = \begin{array}{c} \cdots \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \cdots \end{array} \quad X_m^n(\alpha) = \begin{array}{c} \cdots \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \cdots \end{array} \quad H = \begin{array}{c} | \\ \square \\ | \end{array}$$

where n and m are the number of inputs and outputs respectively, and $\alpha \in [0, 2\pi)$ is an angle called *phase*. If $\alpha = 0$ it will be omitted from the diagram.

We define the semantics of diagrams via an interpretation functor $\llbracket \cdot \rrbracket : \mathcal{D} \rightarrow \mathbf{FdHilb}_{\text{wp}}$, where $\mathbf{FdHilb}_{\text{wp}}$ is the category of complex Hilbert spaces and linear maps under the equivalence relation $f \equiv g$ iff there exists θ such that $f = e^{i\theta} g$. A diagram $f : n \rightarrow m$ output defines a linear map $\llbracket f \rrbracket : \mathbb{C}^{\otimes 2n} \rightarrow \mathbb{C}^{\otimes 2m}$ as follows:

$$\llbracket Z_m^n(\alpha) \rrbracket = \begin{cases} |0\rangle^n & \mapsto |0\rangle^m \\ |1\rangle^n & \mapsto e^{i\alpha} |1\rangle^m \end{cases}$$

$$\llbracket X_m^n(\beta) \rrbracket = \begin{cases} |+\rangle^n & \mapsto |+\rangle^m \\ |-\rangle^n & \mapsto e^{i\beta} |-\rangle^m \end{cases}$$

$$\llbracket H \rrbracket = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The map $\llbracket \cdot \rrbracket$ extends in the evident way to a monoidal functor. We can now see where the name ZX-calculus comes from: the Z vertices are defined in terms of the Z basis of \mathbb{C}^2 while the X vertices are defined in terms of the X basis.

The interpretation of \mathcal{D} contains a universal set of quantum gates. Note that $Z_1^1(\alpha)$ and $X_1^1(\alpha)$ are the rotations around the X and Z axes, and in particular when $\alpha = \pi$ they yield the Pauli X and Z matrices. The $\wedge Z$ is defined by:

$$\wedge Z = \begin{array}{c} | \\ \bullet \quad \square \quad \bullet \\ | \end{array}$$

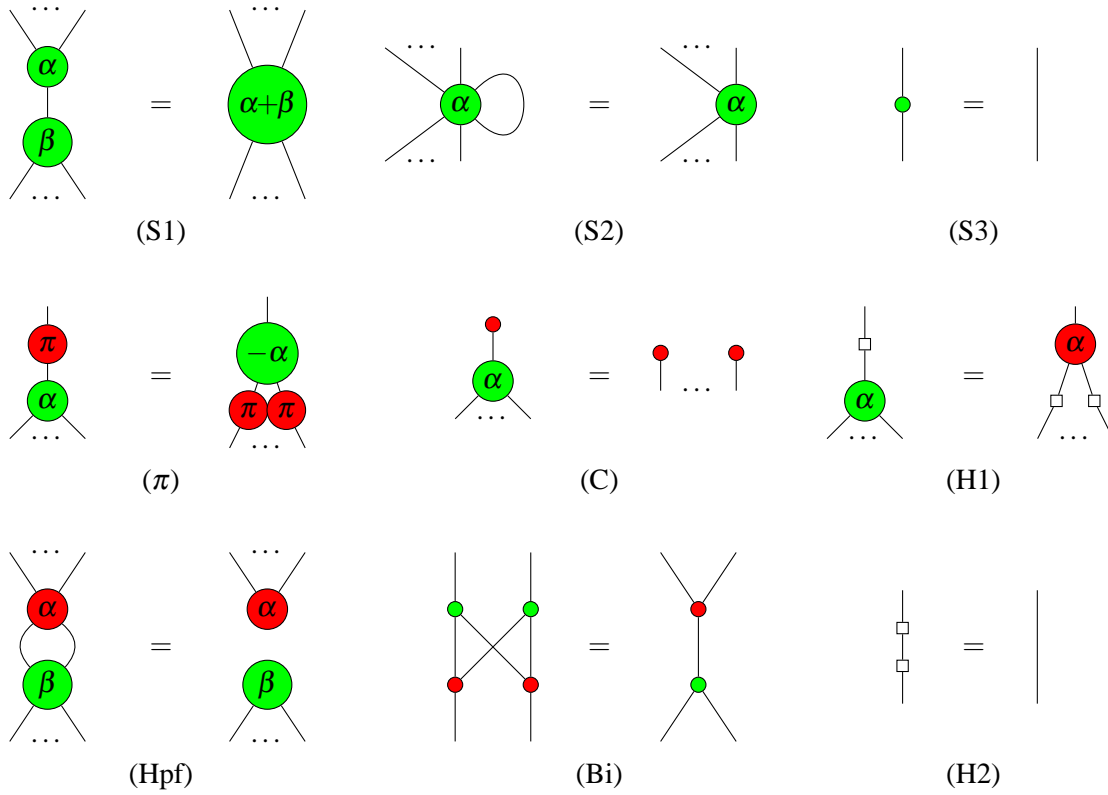
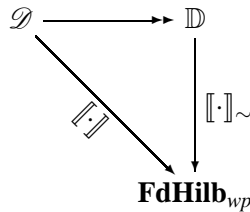


Figure 1: Equations for the ZX-calculus

In order to obtain the ZX-calculus we quotient the free category \mathcal{D} by the equations shown in Figure 1; the quotient category we denote by \mathbb{D} .

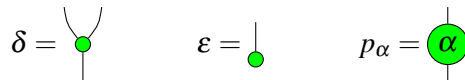
The equations of Fig 1 are sound with respect to the interpretation functor $[[\cdot]]$ introduced above.

Proposition 1.3. *There exists a canonical functor $[[\cdot]]_{\sim} : \mathbb{D} \rightarrow \mathbf{FdHilb}_{wp}$ making the following diagram commute:*



In the rest of this paper we won't make any distinction between \mathcal{D} and \mathbb{D} , nor between the interpretation functors. Indeed, we will abuse notation and refer to both as $[[\cdot]]$.

Remark. Note that in the presence of the equations (S1)-(S3), which we refer to collectively as the “spider rule”, we could have made other choices for the generators of the X and Z families of vertices. For example, the prime graphs



are used in the formulation that emphasises the fact that each family forms a Frobenius algebra. From that perspective the spider rule is effectively a normal-form theorem; see [4] for details.

Proposition 1.4. *The following are direct consequences of the axioms.*

- Any connected diagram containing only Z or only X vertices is equivalent to a prime graph.
- Any diagram without any H is equivalent to a simple bipartite graph.
- Any diagram is equivalent to (a) a diagram with no Z vertices; and (b) a diagram with no X vertices.
- Any equation which holds between two graphs, also holds with Z and X exchanged.

Remark. Note that although Figure 1 seems to favour one colour over the other, by the last point of Proposition 1.4 we know that all the rules apply with the colours reversed.

Euler decomposition of H . The following axiom is not part of the definition of the ZX-calculus


(EU)

In [8] we proved that the Euler decomposition cannot be derived from the axioms of the ZX-calculus; however, in that paper we considered slightly weaker axioms. It is straight-forward to give a counter-model for the ZX-calculus of today.

Lemma 1.5. *The Euler decomposition of H cannot be derived by the rules of the ZX calculus.*

Proof. We define an alternative interpretation functor $\llbracket \cdot \rrbracket_0 : \mathcal{D} \rightarrow \mathbf{FdHilb}_{\text{wp}}$ by

$$\begin{aligned} \llbracket H \rrbracket_0 &= \llbracket H \rrbracket \\ \llbracket Z_m^n(\alpha) \rrbracket_0 &= \llbracket Z_m^n(0) \rrbracket \\ \llbracket X_m^n(\beta) \rrbracket_0 &= \llbracket X_m^n(0) \rrbracket. \end{aligned}$$

It's easy to verify that all the equations of Figure 1 still hold under $\llbracket \cdot \rrbracket_0$ but (EU) fails. □

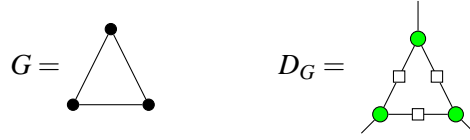
2 Graph states and Local complementation

Definition 2.1. Let $G = (V, E)$ be an undirected graph. Then the *graph state* $|G\rangle$ is defined by

$$|G\rangle = \left(\prod_{uv \in E} \wedge Z_{uv} \right) \bigotimes_{v \in V} |+\rangle$$

Given a graph G we can directly write down the diagram D_G such that $\llbracket D_G \rrbracket = |G\rangle$ as follows: (1) for each $v \in V$ we add a Z vertex, connected to an output; (2) for each edge $uv \in E$ we add an H vertex, connected to those Z vertices corresponding to the vertices u and v .

Example 2.2. Consider the case when G is just a triangle:

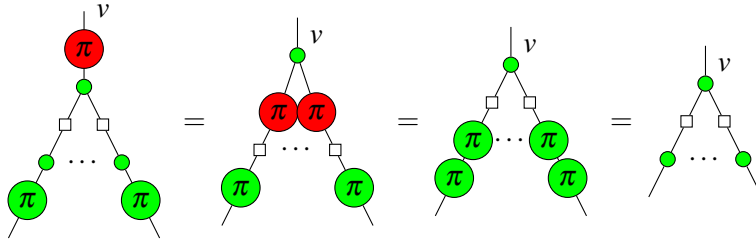


Proposition 2.3. Let $G = (V, E)$ be a graph with $v \in V$ and define

$$K_v = \left(\prod_{u \in N(v)} Z_u \right) X_v.$$

Then $K_v |G\rangle = |G\rangle$.

Proof. We apply an $X(\pi)$ to the output corresponding to v and a $Z(\pi)$ on all the outputs of the neighbours of v :



□

Definition 2.4. Suppose $G = (V, E)$ is a graph with $v \in V$. Let $E_1 = E \cap (N(v) \times N(v))$ and $E_2 = (N(v) \times N(v)) \setminus E_1$. Then the local complementation of G at v is defined by

$$G * v = (V, (E \setminus E_1) \cup E_2).$$

Equivalently, if u, u' are neighbours of v then uu' is an edge of $G * v$ if and only if it is not an edge of G ; otherwise the two graphs are the same.

For graph states local complementation can also be expressed in terms of a product of single qubit operations:

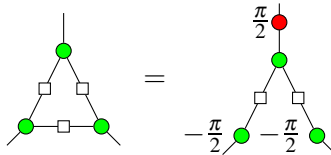
Proposition 2.5 ([14]). Let G be a graph with vertex v ; define

$$M_v = \left(\prod_{u \in N(v)} Z(-\pi/2)_u \right) \cdot X(\pi/2)_v$$

Then $|G * v\rangle = M_v |G\rangle$.

Note that $M_v^2 = K_v$ hence local complementation is involutive on graph states.

Example 2.6. Here we consider the local complementation of the triangle by its top vertex:



Theorem 2.7 ([8]). Proposition 2.5 is equivalent to Equation (EU) in the ZX-calculus, hence it cannot be proven in the ZX-calculus.

3 Pivoting

Pivoting, also known as edge-local complementation, is a local transformation of graphs. Given a graph G with an edge uv , $G \wedge uv$, the graph obtained by pivoting according to uv , consists in exchanging the two vertices u and v and in complementing the tripartite subgraph formed by (i) the common neighbours of u and v ; (ii) the exclusive neighbours of u ; and (iii) the exclusive neighbours of v (see Figure 2).

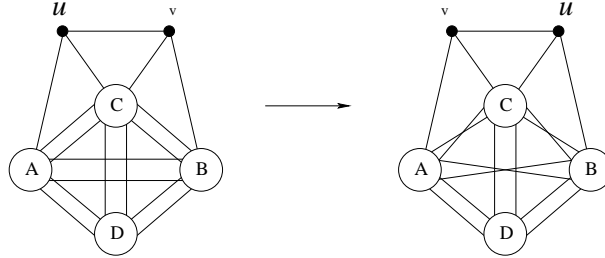


Figure 2: Pivoting on uv . $C = N(u) \cap N(v)$, $A = N(u) \setminus C$, $B = N(v) \setminus C$, and D is the rest of the vertices. Pivoting on uv exchanges vertices u and v , and for any $(x, y) \in (A \times B) \cup (B \times C) \cup (A \times C)$, the edge xy is deleted if xy was an edge, and added otherwise.

Pivoting is a combination of local complementations, $G \wedge uv = G * u * v * u$ (Notice that $G * u * v * u = G * v * u * v$) and can be performed on graph states by applying Hadamard on vertices u and v and Z on their common neighbours:

Proposition 3.1 (*Pivoting Property* [13, 12]).

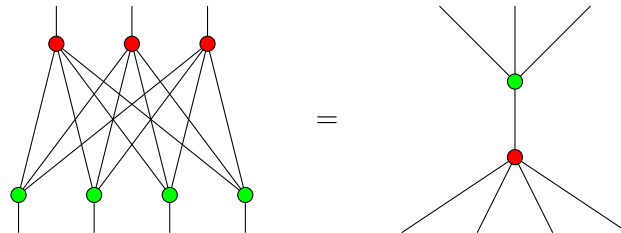
$$|G \wedge uv\rangle = H_{u,v} Z_{N(u) \cap N(v)} |G\rangle$$

Pivoting of graph states have several applications in quantum information processing. In particular the universality of the triangular grid as a resource of measurement-based quantum computing has been proved using pivoting [12]; pivoting can also be used to compute the minimal distance of linear codes [5].

In the rest of this section, we prove that an additional axiom, strictly weaker than the Euler decomposition of H , needs to be added to the ZX-calculus to prove the pivoting property. However, when u and v have no common neighbours, the pivoting property can be proved in the plain ZX-calculus:

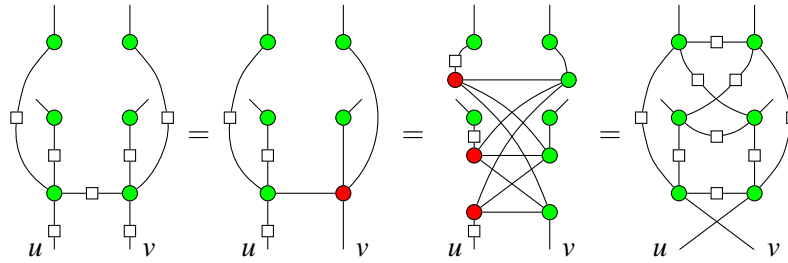
Lemma 3.2. *For any graph $G = (V, E)$ and any $u, v \in V$ which have no common neighbour, $|G \wedge uv\rangle = H_{u,v} |G\rangle$ can be derived in the ZX-calculus.*

Proof. The proof is based on the generalised bialgebra law [8]:



Assume for the moment that there is no edge between the neighbours of u and the neighbours of v . In that case applying H on both u and v permutes u and v and creates a complete bipartite graph between

the neighbours of u and this of v . For instance, if u and v both have two neighbours:



The first equation is via rule (H1) while the second follows from the generalised bialgebra. The final equation uses rule (H1) again to remove all the red vertices, followed by the spider rule, and some rearrangement of the graph. Now suppose that in fact there were some edges between the neighbours of u and those of v . The procedure above will add an additional edge, and then both may be removed since $\square \text{---} \bullet \text{---} \bullet = \bullet \text{---} \bullet$ by (HpF). \square

As a consequence, pivoting of triangle-free graphs or bipartite (or 2-colourable) graphs can be derived in the ZX-calculus. Notice that the pivoting preserves bipartiteness (but not triangle freeness), so one can prove a series of pivotings on bipartite graphs in the ZX-calculus.

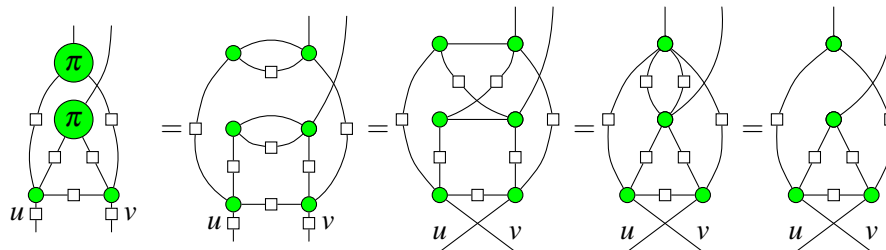
In the following we prove that the pivoting of arbitrary graph can be derived in the ZX-calculus augmented with a new rule for π -rotations:

Theorem 3.3. *Pivoting of arbitrary graph can be proved in the ZX-calculus augmented with the following axiom:*

$$\pi = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \\ \bullet \end{array} \quad (\text{HL})$$

This new axiom is called the *H-loop* axiom as it can be rewritten as $\pi = \bullet \text{---} \square$.

Proof. We illustrate the proof on the particular case where the u and v have two common neighbours:

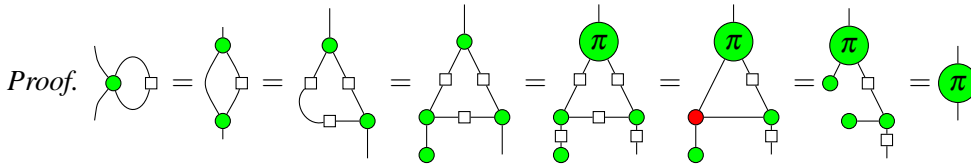
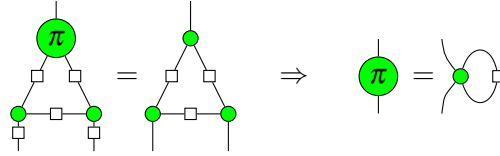


The left-most diagram corresponds to a graph state on which H is applied on two vertices u and v and Z (green π -rotation) on their two common neighbours. The second diagram is obtained using the (HL) axiom. This transformation splits each common neighbour of u and v in such a way that Lemma 3.2 can be applied, leading to the third diagram. The application of the spider rule (fourth diagram) and the Hopf law (fifth diagram) completes the proof for this particular graph.

The general case is similar. First, the π -rotations on the common neighbours are removed using the (HL) axiom, which splits the common neighbours. Then, in the absence of common neighbours Lemma 3.2 is used. Finally spiders and the Hopf Law complete the proof. \square

In the following, we show that one can derive pivoting if and only if Equation (HL) holds:

Lemma 3.4. *In the ZX-calculus, the pivoting property for the triangle implies that the π -rotation is equivalent to a “H-loop”, i.e.*



□

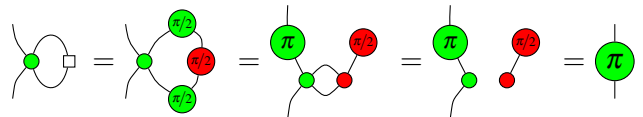
Lemma 3.5. $\pi = \text{H-loop}$ cannot be derived from the rules of the ZX-calculus.

Proof. We consider the interpretation functor $\llbracket \cdot \rrbracket_0$ introduced in Lemma 1.5, which preserves all the axioms of the ZX-calculus, but for which we have:

$$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \llbracket \text{H-loop} \rrbracket_0 \neq \llbracket \pi \rrbracket_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

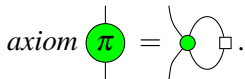
□

Like the Euler decomposition of H (Equation (EU)), Equation (HL) cannot be derived from the rules of the ZX-calculus. The completeness for the stabilisers of the ZX-calculus augmented with Euler decomposition of H guarantees that equation (HL) can be derived from the Euler decomposition of H . Indeed,

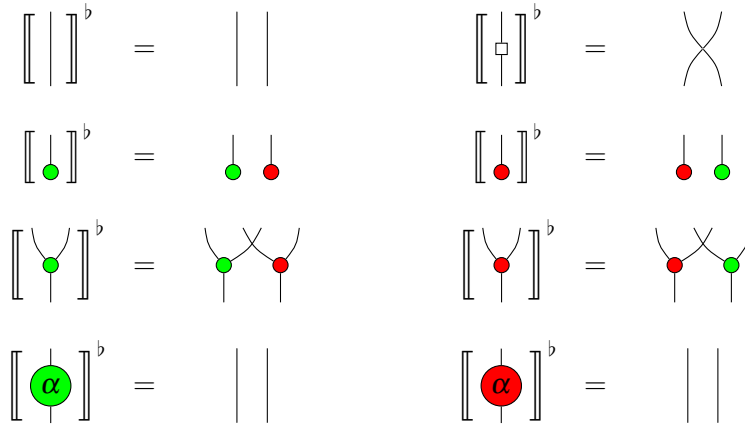


In the following, we prove that Equation (HL) is actually strictly weaker than the Euler decomposition in the sense that the Euler decomposition cannot be derived from Equation (HL) in the ZX-calculus.

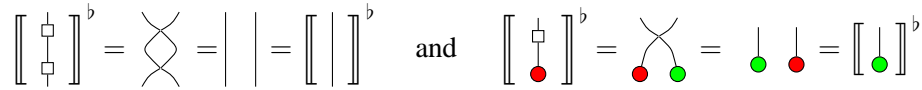
Lemma 3.6. *The Euler decomposition of H cannot be derived in the ZX-calculus augmented with the axiom*



Proof. We consider the following functor $[[\cdot]]^b$ which maps diagrams to diagrams:

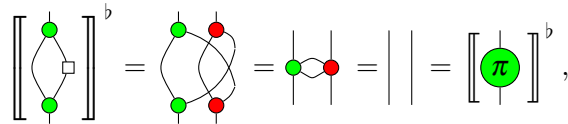


Notice that the axioms of the ZX-calculus are satisfied. Indeed, for diagrams without H , the functor $[[\cdot]]^b$ consists in doubling the picture and trivialising the rotations. Regarding the axioms which involve H , we have

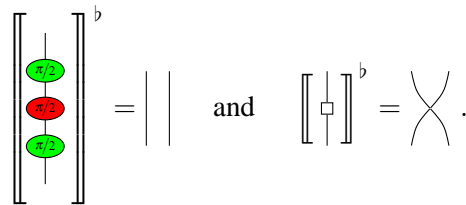


for instance, the other ones are satisfied similarly.

The H-loop axiom is satisfied as well:



but the Euler decomposition is not:



□

The combination of Lemmas 3.5 and 3.6 proves that “ZX-calculus + H-loop” is indeed an intermediate theory between the ZX-calculus and “ZX-calculus + Euler”.

4 Angle-free calculus for Real Stabilizers

Backens [1] considered a syntactic restriction on the terms of the ZX-calculus: by demanding that all the phases occurring in a term are multiples of $\frac{\pi}{2}$, the resulting ZX-calculus terms are in exact correspondence with stabilizer states. Furthermore, the theory of ZX-calculus + Euler is sufficient to decide the equality for these states. In other words, the theory is complete for stabilizer quantum mechanics.

We will now consider a stronger syntactic restriction, namely that all phases must be either 0 or π . Semantically this yields the real-valued fragment of stabilizer quantum mechanics. We will also modify the axiom scheme by dropping the axioms (π) and (C) and replacing them with

$$\begin{array}{ccc}
 \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \bullet \\ | \\ \dots \end{array} \quad \dots \quad \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} \\
 \text{(C1)} & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ | \\ \dots \end{array} \quad \dots \quad \begin{array}{c} \pi \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} \\
 \text{(C2)} & &
 \end{array}$$

Note that these equations are both derivable in the full ZX-calculus. The resulting system we call the weak ZX-calculus.

Lemma 4.1. *The following equations are derivable in the weak ZX-calculus:*

$$\begin{array}{ccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \pi \quad \pi \\ \dots \end{array}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \pi \\ \bullet \\ | \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ | \\ \pi \\ \bullet \\ | \\ \dots \end{array}
 \end{array}$$

Proof. For the first equation we have

$$\begin{array}{ccccccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ | \quad | \\ \dots \quad \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ | \quad | \\ \dots \quad \dots \end{array} & = & \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \pi \quad \pi \\ \dots \end{array}
 \end{array}$$

by spider, (B), (C2), and spider. Making use of this equation, in both its original and colour-switched form, we have:

$$\begin{array}{ccccccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \pi \\ \bullet \\ | \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ | \\ \pi \\ \bullet \\ | \\ \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ / \quad \backslash \\ \pi \quad \pi \\ | \quad | \\ \dots \quad \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ / \quad \backslash \\ \pi \quad \pi \\ | \quad | \\ \dots \quad \dots \end{array} & = & \begin{array}{c} \pi \\ \bullet \\ | \\ \pi \\ \bullet \\ | \\ \dots \end{array}
 \end{array}$$

where the scalar factor was dropped at the last step. □

However, in the presence of Equation (HL) there is no need for the angle π at all. We can now define the “angle-free ZX-calculus” by replacing all the π vertices with loops:

$$\begin{array}{ccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \dots \end{array} & \mapsto & \begin{array}{c} \square \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array}
 \end{array}
 \qquad
 \text{and}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \pi \\ \bullet \\ | \\ \dots \end{array} & \mapsto & \begin{array}{c} \square \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array}
 \end{array}$$

and replacing axiom (C2) with (L):

$$\begin{array}{ccc}
 \begin{array}{c} \square \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} & = & \begin{array}{c} \square \\ \bullet \\ | \\ \dots \end{array} \quad \dots \quad \begin{array}{c} \square \\ \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} \\
 \text{(L)} & &
 \end{array}$$

Evidently, the resulting calculus is strictly stronger than the weak ZX-calculus and weaker than the restricted ZX-calculus + Euler considered by Backens.

We will show that the angle-free ZX-calculus is complete for real-valued stabilizers.

4.1 Real stabilizer quantum mechanics

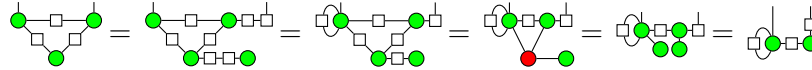
Recall that the Clifford operations are the normalisers of the Pauli operators, i.e. $C_n = \{U \mid \forall g \in P_n, UgU^\dagger \in P_n\}$ where P_n is the Pauli group on n qubits. The *real* Cliffords—i.e. those satisfying $\bar{U} = U$ —form a subgroup of C_n generated by $\{Z, H, \wedge Z\}$. We call *real stabilizer quantum mechanics* any quantum evolution that can be described by real Clifford operations, $|0\rangle$ initialisations, and $|0\rangle$ projections. Notice that the image of the angle-free ZX-calculus under the functor $[\![\cdot]\!]$ coincides with real stabiliser quantum mechanics. We now show the completeness of the angle-free ZX-calculus for real stabiliser quantum mechanics, i.e. for any two diagrams D_1 and D_2 , if $[\![D_1]\!] = [\![D_2]\!]$ then $D_1 = D_2$ can be proved in the calculus.

We follow the proof of the completeness of the ZX-calculus together with the Euler decomposition for (complex) stabiliser quantum mechanics [1]. Due to the Choi-Jamioilkowski isomorphism it suffices to consider input-free diagrams (since any input can be turned into an output). A diagram with no input is called a *diagram state*.

Definition 4.2. A diagram is called a *GS-RLC diagram* if it consists of a graph state with arbitrary single real Clifford operator applied on each output.

Lemma 4.3. Any angle-free diagram state is equal to some GS-RLC diagram within the angle-free ZX-calculus.

Proof Sketch. The proof is by induction. Intuitively, every red dot can be turned into a green dot using H ; the spider rule is used to merge green dots connected by a wire; parallel H -edges are removed using the Hopf law. If there is a green dot which is not connected to an output, then either this dot is disconnected from the rest of the diagram and can be ignored, or the dot can be removed by pivoting with one of its neighbours as shown:



That is, the bottom dot is removed by pivoting along one of its incident edges. □

Definition 4.4. A *reduced* GS-RLC diagram, is a GS-RLC diagram such that

- (1) every vertex Clifford operator is one of $\begin{array}{c} | \\ \square \\ | \end{array}$, $\begin{array}{c} | \\ \square \\ | \\ \bullet \end{array}$, $\begin{array}{c} | \\ \square \\ | \\ \square \end{array}$ or $\begin{array}{c} | \\ \square \\ | \\ \square \\ \bullet \end{array}$
- (2) two adjacent vertices must not both have vertex operators that include an H .

Lemma 4.5. Any angle-free diagram state is equal to a reduced GS-RLC diagram.

Proof Sketch. Any real local Clifford is a combination of H , X and Z . Notice that using Proposition 2.3, every X can be transformed in Z s on its neighbours. As a consequence the vertex Clifford operators are either I , Z , H or HZ . Moreover, if two adjacent vertices have a vertex operator which include an H , then one can do a pivoting which is consuming the H s, transforms the graph and produces Z on the common neighbours. □

Suppose that a pair of GS-RLC diagrams describe states with the same number of qubits, that is, they have the same set of outout vertices. Such a pair is called *simplified* if there is no pair of qubits u and v which are adjacent in at least one diagram and such that H is applied on u but not v in the first diagram, and on v but not u in the second diagram.

Lemma 4.6. Any pair of angle-free diagrams of reduced GS-LRC diagrams can be simplified.

Proof Sketch. If there exists a pair u, v which are adjacent in the first diagram such that H is applied on u and not on v , then one can apply a pivoting on u, v in this graph. This pivoting consumes the H on u and add an H on v . This transformation does not introduce nor remove H on the other vertices, so this transformation can be applied inductively to any pair of vertices which do not satisfies the conditions of simplified pairs of GS-RLC. \square

Theorem 4.7. *Given two reduced GS-RLC diagrams D_1 and D_2 which form a simplified pair, $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ if and only if D_1 and D_2 are identical.*

Proof. Since D_1 and D_2 are reduced GS-RLC diagrams, there exist two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, and four subsets $A_1, A_2, B_1, B_2 \subseteq V$ such that $\llbracket D_i \rrbracket = H_{A_i} Z_{B_i} |G_i\rangle$, where $H_A = \bigotimes_{u \in A} H_u$. Diagrams D_1 and D_2 are identical iff $A_1 = A_2, B_1 = B_2$ and $G_1 = G_2$. First we show that $A_1 = A_2$. Notice that $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $H_A Z_{B_1} |G_1\rangle = Z_{B_2} |G_2\rangle$ where $A = A_1 \Delta A_2$ is the symmetric difference of A_1 and A_2 . By contradiction, for any $u \in A_1 \Delta A_2$, $|G_1\rangle$ is a fix point of $X_u Z_{N_{G_1}(u)}$, so $Z_{B_1} |G_1\rangle$ is an eigenvector of $X_u Z_{N_{G_1}(u)}$. Moreover, since D_1 is in a reduced form there is no H applied on qubits adjacent to u , so $H_A Z_{B_1} |G_1\rangle$ is an eigenvector of $Z_u Z_{N_{G_1}(u)}$. Indeed $H_A Z_{B_1} |G_1\rangle = H_A Z_{B_1} X_u Z_{N_{G_1}(u)} |G_1\rangle = \pm Z_u Z_{N_{G_1}(u)} H_A Z_{B_1} |G_1\rangle$. Regarding the second state, $Z_{B_2} |G_2\rangle$ is an eigenvector of $X_u Z_{N_{G_2}(u)}$. The two operator $X_u Z_{N_{G_2}(u)}$ and $Z_u Z_{N_{G_1}(u)}$ are anti commuting so they cannot have a common non-zero eigenvector, as a consequence $A_1 = A_2$. Thus $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ implies $Z_{B_1} |G_1\rangle = Z_{B_2} |G_2\rangle$. Moreover, it has been proved (Lemma 3 in [12]) that $Z_{B_1} |G_1\rangle = Z_{B_2} |G_2\rangle$ implies $B_1 = B_2$ and $G_1 = G_2$. As a consequence the two diagrams are identical. \square

5 Conclusion and Perspectives

We have introduced a new calculus, intermediate between the ZX-calculus and the ZX-calculus augmented with the Euler decomposition of H . As the introduction of the Euler decomposition was driven by local complementation, the new axiom we consider, namely the H-loop, is driven by another graph transformation, namely pivoting. We prove the H-loop axiom cannot be derived in the plain ZX-calculus, and is strictly weaker than the Euler decomposition of H . When restricted to 0- and π -rotations this new calculus is complete for real stabiliser quantum mechanics. Moreover this restricted language admits a simple equivalent angle-free calculus. We believe this angle-free calculus will be the cornerstone for an axiomatisation of real quantum mechanics. Real quantum mechanics is known to be universal for quantum computing, moreover the restriction to the real field provides some useful simplifications in terms of diagrammatic quantum mechanics (for example, the object A and its dual A^* have the same interpretation). Another example is that, when restricted to real numbers, the unbiased bases are perfectly captured by the complementary observables X and Z of the ZX-calculus, whereas the axiomatisation of the third (complex) mutually unbiased base for qubit, albeit possible (see [11]) is less intuitive. On the other hand applications which require complex numbers like local tomography cannot be captured by this intermediate language, and require additional axioms (e.g. Euler decomposition of H).

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References

- [1] Miriam Backens (2014): *The ZX-calculus is complete for stabilizer quantum mechanics*. *New J. Phys.* 16(093021), doi:10.1088/1367-2630/16/9/093021.
- [2] Ethan Bernstein & Umesh Vazirani (1997): *Quantum complexity theory*. *SIAM Journal on Computing* 26(5), pp. 1411–1473, doi:10.1137/S0097539796300921.
- [3] Bob Coecke & Ross Duncan (2011): *Interacting Quantum Observables: Categorical Algebra and Diagrammatics*. *New J. Phys* 13(043016), doi:10.1088/1367-2630/13/4/043016.
- [4] Bob Coecke, Eric Oliver Paquette & Dusko Pavlovic (2010): *Classical and quantum structuralism*. In S. Gay & I. Mackie, editors: *Semantic Techniques in Quantum Computation*, chapter 2, Cambridge University Press, pp. 29–69.
- [5] Lars Eirik Danielsen & Matthew G Parker (2008): *Edge local complementation and equivalence of binary linear codes*. *Designs, Codes and Cryptography* 49(1-3), pp. 161–170, doi:10.1007/s10623-008-9190-x.
- [6] Ross Duncan (2006): *Types for Quantum Computing*. Ph.D. thesis, Oxford University.
- [7] Ross Duncan (2013): *A graphical approach to measurement-based quantum computing*. In Chris Heunen, Mehrnoosh Sadrzadeh, & Edward Grefenstette, editors: *Quantum Physics and Linguistics: A Compositional, Diagrammatic Discourse*, chapter 3, Oxford University Press, doi:10.1093/acprof:oso/9780199646296.003.0003.
- [8] Ross Duncan & Simon Perdrix (2009): *Graph States and the necessity of Euler Decomposition*. In K. Ambos-Spies, B. Löwe & W. Merkle, editors: *Computability in Europe: Mathematical Theory and Computational Practice (CiE'09), Lecture Notes in Computer Science 5635*, Springer, pp. 167–177, doi:10.1007/978-3-642-03073-4.
- [9] Ross Duncan & Simon Perdrix (2010): *Rewriting measurement-based quantum computations with generalised flow*. In S. Abramsky, C. Gavaille, C. Kirchner, F. Meyer auf der Heide & P. G. Spirakis, editors: *Automata, Languages and Programming, 37th International Colloquium, ICALP 2010, Proceedings Part II, Lecture Notes in Computer Science 6199*, Springer, pp. 285–296, doi:10.1007/978-3-642-14162-1_24.
- [10] André Joyal & Ross Street (1991): *The Geometry of Tensor Categories I*. *Advances in Mathematics* 88, pp. 55–113, doi:10.1016/0001-8708(91)90003-P.
- [11] Alex Lang & Bob Coecke (2011): *Trichromatic Open Digraphs for Understanding Qubits*. In: *Proceedings of 8th Workshop on Quantum Physics and Logic (QPL 2011), Electronic Proceedings in Theoretical Computer Science* 95, pp. 193–209, doi:10.4204/EPTCS.95.14.
- [12] Mehdi Mhalla & Simon Perdrix (2013): *Graph States, Pivot Minor, and Universality of (X,Z)-Measurements*. *IJUC* 9(1-2), pp. 153–171.
- [13] Maarten Van den Nest & Bart De Moor (2005): *Edge-local equivalence of graphs*. *arXiv preprint math/0510246*.
- [14] Maarten Van den Nest, Jeroen Dehaene & Bart De Moor (2004): *Graphical description of the action of local Clifford transformations on graph states*. *Physical Review A* 69, p. 022316, doi:10.1103/PhysRevA.69.022316.
- [15] Peter Selinger (2005): *Dagger compact closed categories and completely positive maps*. In: *Proceedings of the 3rd International Workshop on Quantum Programming Languages*.